

On the $\mathcal{N} = 1^*$ Gauge Theory on a Circle and Elliptic Integrable Systems

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ABSTRACT : We continue our study of the $\mathcal{N} = 1^*$ supersymmetric gauge theory on $\mathbb{R}^{2,1} \times S^1$ and its relation to elliptic integrable systems. Upon compactification on a circle, we show that the semi-classical analysis of the massless and massive vacua depends on the classification of nilpotent orbits, as well as on the conjugacy classes of the component group of their centralizer. We demonstrate that semi-classically massless vacua can be lifted by Wilson lines in unbroken discrete gauge groups. The pseudo-Levi subalgebras that play a classifying role in the nilpotent orbit theory are also key in defining generalized Inozemtsev limits of (twisted) elliptic integrable systems. We illustrate our analysis in the $\mathcal{N} = 1^*$ theories with gauge algebras $su(3)$, $su(4)$, $so(5)$ and for the exceptional gauge algebra G_2 . We map out modular duality diagrams of the massive and massless vacua. Moreover, we provide an analytic description of the branches of massless vacua in the case of the $su(3)$ and the $so(5)$ theory. The description of these branches in terms of the complexified Wilson lines on the circle invokes the Eichler-Zagier technique for inverting the elliptic Weierstrass function. After fine-tuning the coupling to elliptic points of order three, we identify the Argyres-Douglas singularities of the $su(3)$ $\mathcal{N} = 1^*$ theory.

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1 Introduction

The infrared dynamics of supersymmetric gauge theories is a rich and fruitful subject. The classification of massless and massive vacua, and the analysis of their symmetry and duality properties are basic features of the theory. For pure $\mathcal{N} = 1$ supersymmetric gauge theory in four dimensions, which is massive in the infrared, we understand the supersymmetric index [1–5], as well as the transformation properties of the vacua under the broken non-anomalous R-symmetry. It is natural to extend the study of the vacua to other $\mathcal{N} = 1$ supersymmetric gauge theories. We recently completed the census of massive vacua in the $\mathcal{N} = 1^*$ theory that arises from mass deformation of the maximally supersymmetric $\mathcal{N} = 4$ theory in \mathbb{R}^4 [6–9]. This is an interesting playground due to duality symmetries inherited by the $\mathcal{N} = 1^*$ theory from the celebrated duality properties of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In the semi-classical count of vacua, nilpotent orbit theory plays a central role, since $sl(2)$ representations solve the F-term equations of motion, and $sl(2)$ embeddings are intimately related to nilpotent orbits through the Jacobson-Morozov theorem [9].

Upon further compactification of the gauge theory on a circle, there is a method to derive a low-energy effective superpotential for the $\mathcal{N} = 1^*$ theory with any gauge group. The method is based on a soft breaking of $\mathcal{N} = 2^*$ supersymmetry by a third mass deformation, as well as the identification of the integrable system for $\mathcal{N} = 2^*$ theory and its quadratic Hamiltonian [10–14]. The effective potential on \mathbb{R}^4 can then be recuperated from the radius independent potential on $\mathbb{R}^{2,1} \times S^1$. However, in this procedure it is clear that one should be mindful about the global distinctions between the gauge theory on \mathbb{R}^4 and the theory on $\mathbb{R}^{2,1} \times S^1$. An example of such subtlety is provided by the supersymmetric index of pure $\mathcal{N} = 1$ theories which indeed depends on those global properties. In that case, the choice of the center of the gauge group and the spectrum of line operators is crucial in computing the vacuum structure after compactification on S^1 [15, 16].

The comparison of semi-classical calculations in $\mathcal{N} = 1^*$ gauge theory to the properties of the corresponding twisted elliptic Calogero-Moser integrable system allows to construct a beautiful bridge between $\mathcal{N} = 1$ gauge theories and integrable systems [11, 14, 17]. The further detailed comparison of the global features of the theory on $\mathbb{R}^{2,1} \times S^1$ will add ornaments to this bridge. In this paper, we argue that upon circle compactification, more intricate features of nilpotent orbit theory come into play. Indeed, the non-trivial topology allows for turning on Wilson lines that can increase the number of massive vacua through various mechanisms [17]. One such mechanism is the presence of a non-trivial component group in the unbroken gauge group. The Wilson lines can then take values in the component group, thus enhancing the number of semi-classical vacua. A second mechanism is the breaking of gauge groups with abelian factors through a Wilson line expectation value [17]. Thus, a classification of nilpotent orbits along with the conjugacy classes of their component groups becomes pertinent. A crucial step in that classification is the listing of pseudo-Levi subalgebras [18–20]. We show that the latter also play a leading

role in listing the semi-classical limits of elliptic integrable systems that generalize the Inozemtsev limit of [21].

We have structured our presentation as follows. Our paper contains advanced nilpotent orbit theory, complexified integrable system analysis, as well as intricate aspects of $\mathcal{N} = 1$ gauge theories in four dimensions upon circle compactification. We have therefore decided to first illustrate many features of the generic analysis in the example of $\mathcal{N} = 1^*$ theory with gauge algebra $so(5)$, where a lot of details can be worked through by hand. We include a description of the consequences of the choice of global gauge group and the spectrum of line operators, which neatly complements the analysis of [15, 16] in an example that is intermediate between $\mathcal{N} = 4$ and pure $\mathcal{N} = 1$ supersymmetric gauge theory in four dimensions. Section 2 serves to study a tree before exploring the forest. The finer features of the $so(5)$ example will motivate the later sections.

In section 3 we make a link between the classification of nilpotent orbits and the conjugacy classes of the component group of their centralizer on the one hand, and limits of elliptic integrable systems on the other. We illustrate features of this analysis in section 4 in the example of the gauge algebra $su(N)$, which will allow to demonstrate the existence of branches of massless vacua, as well as the use of generalized Inozemtsev limits. We will obtain an explicit analytic description of the massless vacua of the $\mathcal{N} = 1^*$ theory with $su(3)$ gauge algebra, and their duality properties. The description of the massless branch in terms of the elliptic system variables invokes intricate aspects of the theory of elliptic functions. The branch of massless vacua has a (Argyres-Douglas) singularity. The singularity also will show up as a point of monodromy for the position of the massless vacua described in terms of the complexified Wilson lines on the torus. The singularity lies at the elliptic point of order three on the boundary of the fundamental domain of the modular group. At the hand of the gauge algebra $su(4)$, we illustrate further aspects that pop up at higher rank.

We also discuss the $\mathcal{N} = 1^*$ theory with gauge group of exceptional type G_2 . A first reason to study this case is that G_2 is a gauge group of limited rank, allowing for an elaborate numerical analysis of the duality properties of the massive vacua. A second reason is that the group G_2 exhibits an orbit with an unbroken discrete gauge group. This will allow us to cleanly illustrate the role played by the discrete group in the identification of the extrema of the integrable system with massive gauge theory vacua on $\mathbb{R}^{2,1} \times S^1$. This aspect puts into focus the difference between the gauge theory on \mathbb{R}^4 and the gauge theory compactified on a circle.

In section 5, we thus provide a large amount of detail of the semi-classical analysis of the vacua of $\mathcal{N} = 1^*$ theory on $\mathbb{R}^{2,1} \times S^1$ with gauge group G_2 , including a nilpotent orbit classification with their pertinent properties, and the low-energy quantum dynamics in the corresponding phases. Moreover, we perform an in-depth analysis of the associated twisted elliptic Calogero-Moser integrable system, and we make a comparison with the semi-classically predicted vacua. We also provide the duality diagram of the massive vacua and a first estimate of a point of monodromy. In section 6, we tie up a loose end, and analytically describe the branch of massless vacua for the $so(5)$ theory. We conclude in section 7 with a summary, and a partial list of open problems on the intersection of supersymmetric gauge

theory, nilpotent orbit theory, integrability, modularity and the theory of elliptic functions.

2 The $\mathcal{N} = 1^*$ Theory with Gauge Algebra $so(5)$

To illustrate finer points that crop up when analyzing $\mathcal{N} = 1^*$ gauge theories with generic gauge group upon circle compactification, we concentrate in this section on the study of $\mathcal{N} = 1^*$ theory with gauge algebra $so(5)$, and the associated twisted elliptic integrable system with root system B_2 [14]. Our analysis in this and the following sections is a continuation of the work presented in [9, 17]. In particular, we refer to [17] for the detailed discussion of the correspondence between the gauge theory and the numerical results on the elliptic integrable system, and we relegate to [9] the full explanation of the relevance of nilpotent orbit theory to the semi-classical gauge theory on \mathbb{R}^4 . We moreover refer to [22–25] for pedagogical introductions to nilpotent orbit theory. We creatively combine these sources in the following.

2.1 The Semi-Classical Analysis and Nilpotent Orbit Theory

The $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on \mathbb{R}^4 has fields in one vector and three chiral multiplet representations of the $\mathcal{N} = 1$ supersymmetry algebra. All fields transform in the adjoint representation of the gauge algebra. After triple mass deformation to $\mathcal{N} = 1^*$ gauge theory, the F-term equations of motion (divided by the complexified gauge group) for the three adjoint chiral scalars have solutions classified by embeddings of $sl(2)$ commutation relations inside the adjoint of the gauge algebra. By the Jacobson-Morozov theorem, these $sl(2)$ triples are in one-to-one correspondence with nilpotent orbits, which have been classified for simple algebraic groups [22–25].

Nilpotent orbits of the classical groups can be enumerated by partitions that correspond to the dimensions of the $sl(2)$ representations that arise upon embedding in the gauge algebra.² The Lie algebra of the centralizer has been computed, and non-abelian centralizers give rise to effective pure $\mathcal{N} = 1$ gauge theories that have a number of quantum vacua equal to the dual Coxeter number of the unbroken gauge group. The partition, the unbroken gauge algebra, and the number of massive quantum vacua they give rise to on \mathbb{R}^4 for the gauge algebra $so(5)$ are enumerated in the first three columns in table 1. For instance, the $2 + 2 + 1$ partition of 5 corresponds to a configuration for the adjoint scalar expectation values that represent a particular orbit (via the correspondence between $sl(2)$ embeddings and nilpotent orbits), and these vacuum expectation values leave a $C_1 = A_1$ gauge algebra unbroken. The resulting pure $\mathcal{N} = 1$ gauge theory at low energy gives rise to two massive vacua. See [6–9, 17].

²For the case of gauge algebra $so(2N)$ and the adjoint gauge group $SO(2N)$, the very even partitions (having only even parts with even multiplicity) give rise to two distinct nilpotent orbits. For this gauge algebra, each orbit gives rise to its own vacua. When the outer automorphism of $so(2N)$ is joined to the adjoint gauge group, we obtain the gauge group $O(2N)$ in which these orbits and the corresponding vacua are identified. See [9] for a detailed discussion.

Orbit Partition	Unbroken	Massive Vacua on \mathbb{R}^4	W-class	Levi
$1 + 1 + 1 + 1 + 1$	B_2	3	\emptyset	0
$2 + 2 + 1$	C_1	2	$\{\alpha_1\}$	C_1
$3 + 1 + 1$	$u(1)$	0	$\{\alpha_2\}$	\tilde{A}_1
5	1	1	$\{\alpha_1, \alpha_2\}$	B_2

Table 1: Nilpotent orbit data for $so(5)$.

The last two columns in table 1 are related to the Bala-Carter theory of nilpotent orbits [18, 19] that associates a Weyl group equivalence class of subsets of the set of simple roots to each Levi subalgebra of the gauge algebra. The reader may revert to studying these columns after reading section 3. See also section 5.2 for Bala-Carter theory with an example worked out in detail. When we compactify the gauge theory on $\mathbb{R}^{2,1} \times S^1$, properties of the centralizer beyond its Lie type become crucial. A refined classification of the nilpotent orbits, including the conjugacy classes of the component group³ of the unbroken gauge group (by Bala, Carter and Sommers [18–20]) gives rise to table 2.

Orbit	Centr.	C. C.	Massive Vac.	W-classes	PLS
$1 + 1 + 1 + 1 + 1$	B_2	1	3	\emptyset	0
$2 + 2 + 1$	A_1	1	2	$\{\alpha_0\}, \{\alpha_1\}$	C_1
$3 + 1 + 1$	$u(1)$	1	0	$\{\alpha_2\}$	\tilde{A}_1
$3 + 1 + 1$	$u(1)$	(12)	1	$\{\alpha_0, \alpha_1\}$	D_2
5	0	1	1	$\{\alpha_0, \alpha_2\}, \{\alpha_1, \alpha_2\}$	B_2

Table 2: The Bala-Carter-Sommers classification of nilpotent orbits with their centralizers, including the conjugacy classes (C.C.) of the component group of the centralizer. The first column gives the partition labeling the orbit, the second the Lie type of the centralizer (i.e. the unbroken gauge algebra for given adjoint vacuum expectation values), the third the conjugacy class of the discrete part of the centralizer corresponding to the chosen pseudo-Levi subalgebra (PLS) in the last column, the fourth the number of massive vacua and the previous to last the Weyl conjugacy classes of subsystems of simple roots of the affine root system. In each case, there is only one distinguished parabolic subalgebra, which is the principal one. This analysis is valid for the adjoint group and will be further refined when we take into account the choice of global properties of the gauge group (see table 3).

At this stage, we wish to take away the elementary fact that the partition $3 + 1 + 1$ appears twice in the first column of table 2, because there is a discrete \mathbb{Z}_2 component subgroup of the centralizer. The \mathbb{Z}_2 component group has two conjugacy classes, namely the trivial one, and the non-trivial one (labeled by the cyclic permutation (12)). The importance of the second occurrence is the fact that we can turn on a Wilson line on the circle equal to this conjugacy class while still satisfying the equations of motion (as discussed in detail in section 5). Because the \mathbb{Z}_2 forms a semi-direct product with the

³The component group is the quotient of the group by its identity component.

$SO(2)$ unbroken gauge group for the $3+1+1$ partition, turning on the Wilson line breaks the abelian gauge group, and generates a new massive vacuum on $\mathbb{R}^{2,1} \times S^1$ [17]. Finally, we note that we also have a massless branch of rank one.

2.2 The Elliptic Integrable System

We turn to how the physics of the $\mathcal{N} = 1^*$ theory with gauge algebra $so(5)$ is coded in the twisted elliptic integrable system of type B_2 that was proposed to be the low-energy effective superpotential for the model [14]. In as far as this constitutes a review of the results presented in [17], we will again be concise, while new features will be emphasized.

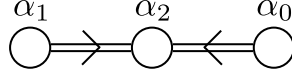


Figure 1: The Dynkin diagram of the affine algebra $\widehat{so(5)} = B_2^{(1)}$ with our convention for the numbering of long and short roots.

The Dynkin diagram for the affine algebra $B_2^{(1)}$ (as well as its finite counterpart, upon deleting the zeroth node) can be read off from figure 1. The long simple root α_1 of B_2 can be parametrized as $\alpha_1 = \epsilon_1 - \epsilon_2$ and the short root α_2 as $\alpha_2 = \epsilon_2$, where the ϵ_i are orthonormal basis vectors in a two-dimensional Euclidean vector space.⁴ The superpotential of the twisted elliptic Calogero-Moser model with root system B_2 is [12]

$$\mathcal{W}_{B_2,tw}(Z) = \wp(z_1 + z_2) + \wp(z_1 - z_2) + \frac{1}{2} [\wp_2(z_1) + \wp_2(z_2)] \quad (2.1)$$

$$= \wp(Z_1) + \wp(Z_1 + Z_2) + \frac{1}{2} \left[\wp_2\left(\frac{Z_2}{2}\right) + \wp_2\left(Z_1 + \frac{Z_2}{2}\right) \right], \quad (2.2)$$

where we combine the Wilson line a and dual photon σ of the low-energy theory in the Coulomb phase in a complex field $Z = \sigma + \tau a$ parametrized by

$$Z = Z_1\pi_1 + Z_2\pi_2 = z_1\epsilon_1 + z_2\epsilon_2. \quad (2.3)$$

Throughout the paper we use capital letters to denote the components of an element of the dual Cartan space decomposed on the basis of fundamental weights, and small letters to denote its components in the basis ϵ_i . For $so(5)$, the relation is

$$\begin{cases} Z_1 = z_1 - z_2 \\ Z_2 = 2z_2 \end{cases} \quad \text{or} \quad \begin{cases} z_1 = Z_1 + \frac{1}{2}Z_2 \\ z_2 = \frac{1}{2}Z_2 \end{cases}. \quad (2.4)$$

The superpotential \mathcal{W} depends on the elliptic Weierstrass function \wp with half-periods $\omega_1 = 1/2$ and $\omega_2 = \frac{\tau}{2}$ (where the complexified gauge coupling is $\tau = \omega_2/\omega_1$) and its twisted cousin \wp_2 which is defined to have half the period in the ω_1 direction, $\wp_2(z, \tau) =$

⁴ Let us recall a few Lie algebra data for future reference. The root lattice is generated by $\epsilon_{1,2}$. The fundamental weights are $\pi_1 = \epsilon_1$ and $\pi_2 = (\epsilon_1 + \epsilon_2)/2$. The dual simple roots are $\alpha_1^\vee = \alpha_1$ and $\alpha_2^\vee = 2\alpha_2 = 2\epsilon_2$. The dual weight lattice is spanned by the ϵ_i . The Weyl group allows for permutations of the ϵ_i , and all sign changes. We follow the conventions of [26].

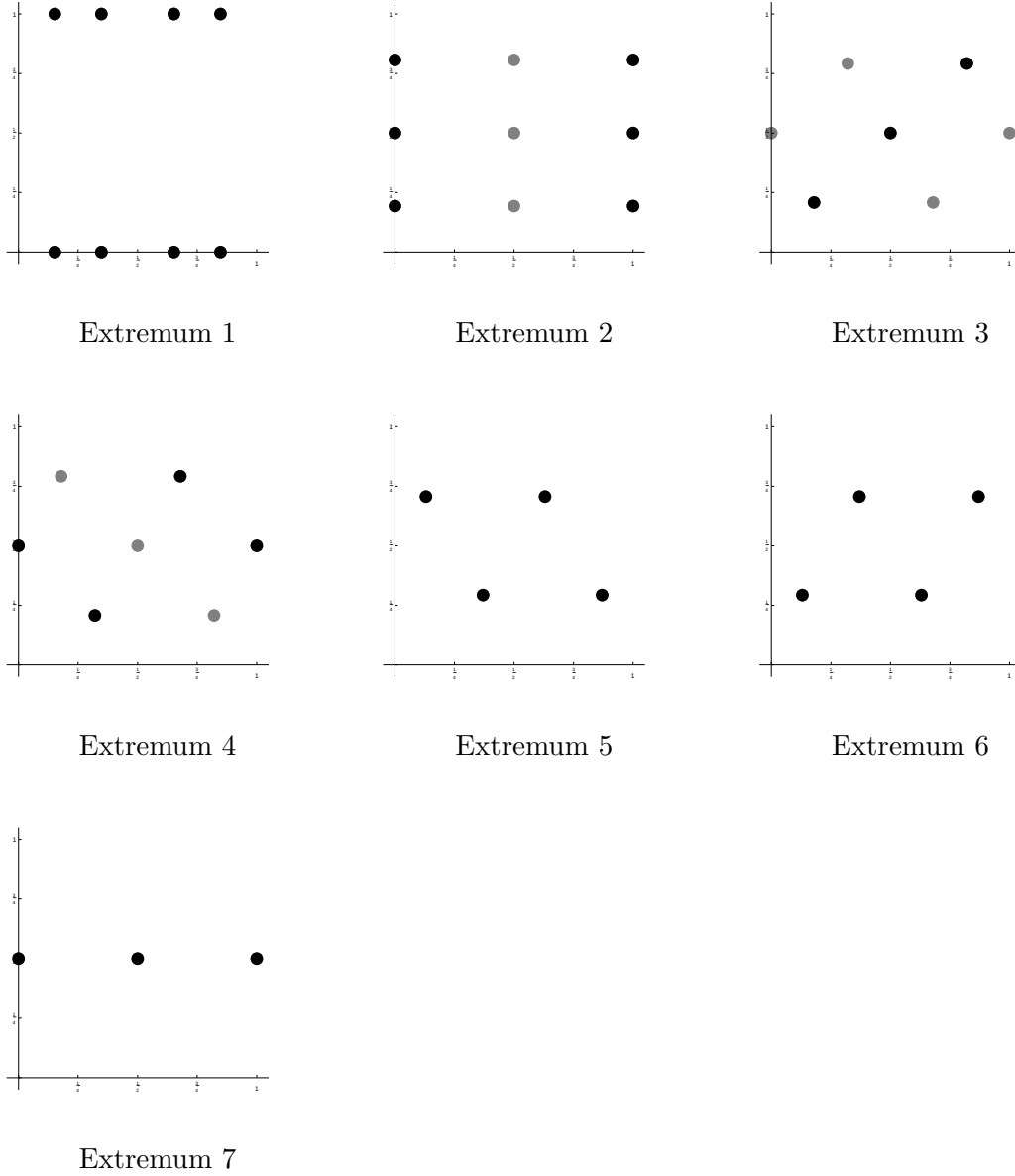


Figure 2: Extrema for the superpotential at coupling $\tau = i$ for the Lie algebra $so(5)$ are drawn in dark. Configurations obtained by translation by ω_1 are drawn in light gray.

$\wp(z, \tau) + \wp(z + \frac{1}{2}, \tau)$. The ratio of the coupling constants for short and long roots was fixed in [14] and checked using Langlands duality in [17]. In [17], we established the existence of seven massive vacua (up to a given equivalence relation to be discussed shortly), determined their positions numerically, and provided analytic expressions for the value of the superpotential in each of these massive vacua. The extremal positions at $\tau = i$ are rendered in figure 2. We moreover established the duality diagram in figure 3 between the seven massive vacua.

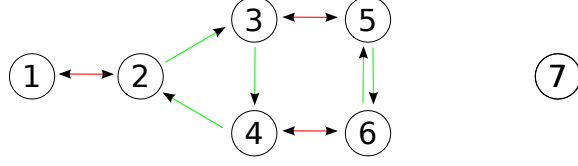


Figure 3: The diagram of dualities on the extrema of the integrable system for $so(5)$. In red, we draw the action of Langlands S_2 -duality, and in green, T -duality, when the action on a given extremum is non-trivial.

In the present section, we wish to add to the analysis presented in [17] in several ways. We analyze the semi-classical limits of the effective low-energy superpotential. We propose a list of such limits, and show that we obtain an analytic handle on each of the seven vacua, and on the massless vacua as well. Moreover, we will carefully exhibit the differences between various global choices of gauge group and spectra of line operators, and consequently a more refined duality diagram.

Importantly, our list of limits is based on table 2. Each nilpotent orbit and conjugacy class of the component group is associated, by Bala-Carter-Sommers theory to a choice of inequivalent⁵ subsystem of simple roots of the affine root system. To each such subsystem, we associate a limit of the integrable system as follows. We demand that for simple roots α_i in the subsystem, we have that the simple root is orthogonal to the vector of extremal positions Z , namely $(\alpha_i^\vee, Z) = 0$, to leading (linear) order in the complexified gauge coupling τ in the large imaginary τ limit. The simple root systems in the complement must have non-zero leading term.

We denote the part in Z that is linear in τ by $Y\tau$. We moreover introduce the redundant coordinate Y_0 which we constrain by the equation $Y_0 + Y_1 + Y_2 = 1$. It is a coordinate that is natural in treating this problem governed by affine algebra symmetry as will become more manifest in section 3. The list of semi-classical limits that we will consider are then labelled by the set J_0 which contains all i for which α_i is in the chosen simple root subsystem indicated in table 2. We therefore distinguish five limits, which we treat one-by-one below.

- The first limit corresponds to the empty set, $J_0 = \emptyset$. The arguments of the Weierstrass functions will all contain a linear term in τ . Therefore all terms are well-approximated by exponentials (see section 3, formulas (3.6) and (3.10)). The limiting procedure in this case is described in detail in [13], which in turn is a generalization of the Inozemtsev limit [21]. By demanding that all these exponentials have the same dependence on the instanton counting parameter $q = e^{2\pi i \tau}$, which is necessary in order to stabilize all variables, we determine that the linear behavior of the coordinates in τ is $Y_0 = Y_1 = Y_2 = 1/3$. We obtain a (fractional instanton) B_2 affine Toda system

⁵The equivalence relation is given precisely in [20], and can be technical in some cases. For the gauge theories we are concerned with, it can be stated as follows. In algebras of type A and G , two subsystems are equivalent if they have the same Lie algebra type and the same repartition of long and short roots. For type B , one should moreover distinguish between $A_1 + A_1$ and D_2 , and between A_3 and D_3 , using the index of the subsystem.

in the limit, with 3 extrema. The solutions (z_1, z_2) in the semi-classical limit are then $(\frac{\tau}{2}, \frac{\tau}{6})$, $(\frac{\tau}{2}, \frac{1}{3} + \frac{\tau}{6})$ and $(\frac{\tau}{2}, \frac{2}{3} + \frac{\tau}{6})$. One can check these solutions against the behavior of the numerical extrema labelled 2, 3 and 4 in [17] (and figures 2 and 3), and they match in the semi-classical limit. This codes the physics of the pure $\mathcal{N} = 1$ gauge theory with gauge algebra $so(5)$.⁶ Indeed, the partition $1 + 1 + 1 + 1 + 1$ leaves the whole of the gauge group unbroken.

- The second case is the choice of subroot system $J_0 = \{0\}$. Note that this is completely equivalent to the choice $J_0 = \{1\}$, since the corresponding marked Dynkin diagrams are the same in both cases, so we concentrate on the first of these sets. Then we have $Y_0 = 0$ as a consequence, and to match powers of q in subleading terms, we choose $Y_1 = Y_2 = 1/2$. In the semi-classical limit, we then obtain a trigonometric A_1 system at leading order (associated to the long root α_0). At subleading order, we find a superpotential $\mathcal{W}(z_1, z_2)$ consisting of a sum of exponentials

$$\mathcal{W}_{B_2, tw} \left(\frac{3\tau}{4} + \frac{1}{4} + \frac{\delta x}{2}, \frac{\tau}{4} + \frac{1}{4} - \frac{\delta x}{2} \right) = \pi^2 (12e^{-2i\pi\delta x} - 4e^{2i\pi\delta x}) q^{\frac{1}{2}} + \dots \quad (2.5)$$

The two extrema at large τ are $(\frac{1}{8} + \frac{3\tau}{4}, \frac{3}{8} + \frac{\tau}{4})$ and $(-\frac{1}{8} + \frac{3\tau}{4}, \frac{5}{8} + \frac{\tau}{4})$.⁷ These match the behavior of the massive vacua number 5 and 6 in figures 2 and 3 at large τ . These are the two confining vacua of the unbroken pure $\mathcal{N} = 1$ $su(2)$ gauge theory. Note how this limit is intermediate in that one coordinate is fixed at leading order in the $q^{\frac{1}{2}}$ expansion, while a second is fixed at subleading order.

- Thirdly, we have the case $J_0 = \{1, 2\}$, which by the same token is equivalent to $J_0 = \{0, 2\}$. We find the trigonometric potential B_2 with a real extremum, which can be characterized in terms of zeroes of orthogonal polynomials [27, 28]. This corresponds to the fully Higgsed vacuum, with label 1 in figure 2. Importantly, there are other, complex extrema of the trigonometric integrable system.⁸ In the limit $\tau \rightarrow i\infty$, one example extremum is given by $(z_1, z_2) \sim (\frac{1}{4} + \frac{\log(1+\sqrt{2})}{2\pi}i, \frac{1}{4} - \frac{\log(1+\sqrt{2})}{2\pi}i)$. This is a massless extremum, part of a branch that we analyze in section 6.
- For our fourth pick, we take $J_0 = \{0, 1\}$ and obtain two trigonometric potentials, corresponding to the root system D_2 . We find the one extremum $(\frac{\tau}{2}, \frac{1}{2} + \frac{\tau}{2})$. This corresponds to extremum number 7. This is a massless vacuum lifted by the presence of a \mathbb{Z}_2 Wilson line. It is thus semi-classically massive on $\mathbb{R}^{2,1} \times S^1$. The \mathbb{Z}_2 Wilson line sits inside the non-trivial conjugacy class (12) of the component group \mathbb{Z}_2 of the unbroken gauge group. This is an occurrence of a general phenomenon that we analyze further in section 5.
- Finally, we turn to the fifth possibility, $J_0 = \{2\}$. The leading τ behavior of the second coordinate is $Y_2 = 0$. As a first stab at the semi-classics in this regime, we

⁶We discuss the global choice of gauge group and line operators in subsection 2.3.

⁷The subleading behavior of the extrema in the large τ limit can easily be computed analytically as well. See later for more intricate explicit examples.

⁸Complex extrema of integrable systems are rarely discussed. The observation we make here on the trigonometric B_2 integrable system, for instance, appears to be new.

choose the values $Y_0 = Y_1 = 1/2$, which is a natural ansatz given the symmetry of the Dynkin diagram about α_2 . In any event, we obtain the trigonometric \tilde{A}_1 system (where the tilde stands for a short root) at leading order. The extremization of the superpotential at order 0 gives $Z_2 = \frac{1}{2}$. The value of Z_2 gets corrected non-perturbatively, namely at order $q^{\frac{1}{2}}$, order $q^{\frac{3}{2}}$ and higher strictly half-integer orders, by terms depending on δZ_1 exponentially. For the particular value of Y_1 that we chose we find

$$\delta Z_2 = \frac{1}{\pi} \left(-e^{-2i\pi\delta Z_1} q^{\frac{1}{2}} - e^{2i\pi\delta Z_1} q^{\frac{1}{2}} + \frac{1}{3} e^{-6i\pi\delta Z_1} q^{\frac{3}{2}} + 5e^{-2i\pi\delta Z_1} q^{\frac{3}{2}} + 5e^{2i\pi\delta Z_1} q^{\frac{3}{2}} + \frac{1}{3} e^{6i\pi\delta Z_1} q^{\frac{3}{2}} + \dots \right). \quad (2.6)$$

Injecting this value for Z_2 in the superpotential $\mathcal{W}(Z_1, Z_2)$ finally gives

$$\begin{aligned} \mathcal{W}_{B_2, tw} \left(\frac{\tau}{2} + \delta Z_1, \frac{1}{2} + \delta Z_2(\delta Z_1) \right) &= \pi^2 \left(\frac{2}{3} + 16q + 16q^2 + 64q^3 + 16q^4 + \dots \right) \\ &= -\frac{2\pi^2}{3} E_{2,2}(q), \end{aligned} \quad (2.7)$$

to order q^4 . We observe that non-perturbatively correcting the leading coordinate Z_2 leads to a vanishing potential for Z_1 , in perturbation theory in q . The value of the coordinate Z_1 determines the non-perturbative correction to the leading coordinate Z_2 . For instance, for the special value $\delta Z_1 = 1/2$, the non-perturbative correction is zero. See equation (2.6). Thus, we find a one-dimensional complex branch of massless vacua to which we return in section 6. The value of the superpotential in these vacua can be determined by a combination of numerics, and analytical expectations to be $\mathcal{W} = -\frac{2\pi^2}{3} E_{2,2}(q)$. The Einstein series $E_{2,2}$ is the modular form of weight 2 of $\Gamma_0(2)$ that has a q -expansion that starts out with -1 .⁹

We have made a list of semi-classical limits for the $so(5)$ integrable system. In particular, we have analytically recuperated all the numerical results of [17], in the large imaginary τ limit. We have moreover made inroads into extra vacua, which are massless. Before discussing the particular features of the $so(5)$ analysis that we will concentrate on in the rest of the paper, we pause to discuss global aspects of the gauge theory at hand.

2.3 Global Properties of the Gauge Group and Line Operators

Up to now, we have implemented a concept of equivalence on the configuration space in which we identify the variables proportional to ω_1 by shifts in the weight lattice and the variables in the ω_2 direction by shifts in the dual weight lattice [17]. These are natural identifications when one is concerned with analyzing the elliptic integrable potential. However, from the gauge theory perspective, the global and local symmetries are fixed a priori, and in this subsection we will carefully track how they influence both the counting of vacua and their duality relations.

⁹See [17] for more details on the combination of techniques used to determine these modular forms.

In other words, we give an example of how to generalize the analysis of the global choice of gauge group and the spectrum of line operators, performed for pure $\mathcal{N} = 1$ gauge theories and $\mathcal{N} = 4$ theories in [15, 16] to $\mathcal{N} = 1^*$ theories. Recall that $\mathcal{N} = 4$ gauge theories with $so(5)$ gauge algebra come in three varieties which satisfy Dirac quantization and maximality of the operator algebra. We first distinguish between the choice of gauge group $SO(5)$ and $Spin(5)$.¹⁰ The $Spin(5)$ theory is unique. The $SO(5)$ theories come in two versions, depending on whether they include a 't Hooft operator which transforms in the fundamental of the dual gauge group, or a Wilson-'t Hooft operator that transforms in the fundamental of both the electric and the magnetic gauge group. The first can be denoted $SO(5)_+$ theory, and the second $SO(5)_-$ theory. The refined duality map of $\mathcal{N} = 4$ theories described in [15, 16] states that the $SO(5)_+$ theory is S_2 -dual to the $Spin(5)$ theory.¹¹ The $SO(5)_-$ theory is self- S_2 -dual. The goal of this subsection is to carefully examine the global electric and magnetic identifications of the extrema of the low-energy effective superpotential to show that the refined classification of vacua of the $\mathcal{N} = 1^*$ theory is consistent with the duality imparted by the $\mathcal{N} = 4$ theory.

To make contact with our set-up, we first analyze the periodicity of the Wilson line, which follows from the global choice of gauge group and line operators. In the case where we work with the adjoint gauge group $Spin(5)/\mathbb{Z}_2 = SO(5)$ and the spectrum of line operators corresponding to the $SO(5)_+$ theory, we allow gauge parameters that close only up to an element in the center of the covering group. The Wilson line periodicity is then the dual weight lattice. The dual weight lattice is spanned by the ϵ_i and therefore the two variables on the Coulomb branch will each have periodicity $2\omega_2$. When the gauge group is the covering group $Spin(5)$, gauge parameters are strictly periodic, and the periodicity of the Wilson line is the dual root lattice. In this case, Wilson lines are equivalent under shifts by $\epsilon_1 - \epsilon_2$ and $2\epsilon_2$. Thus, both coordinates are periodic with periodicity $4\omega_2$, and we can further divide by simultaneous shifts by $2\omega_2$.

For the magnetic line operator spectrum for the $Spin(5)$ and $SO(5)_+$ theories, it suffices to Langlands S_2 -dualize the above reasoning. We thus obtain that for $SO(5)_+$ we can shift by $2\omega_1$ separately each coordinate (i.e. by the root lattice), and for $Spin(5)$ we add on top of this the simultaneous shift by ω_1 (i.e. the weight lattice). The factor of two difference in the lattice spacing is due to the mechanics of the Langlands S_2 duality. For the $SO(5)_-$ theory, the story is more subtle. There is a 't Hooft-Wilson line operator in the spectrum which is in the fundamental of both the dual gauge group and the ordinary gauge group. We allow for the identifications common to $Spin(5)$ and $SO(5)_+$, and add the identification that shifts an individual coordinate by $2\omega_2$ and both coordinates simultaneously by ω_1 . This is the diagonal \mathbb{Z}_2 in the magnetic and electric weight lattices divided by the magnetic and electric root lattices respectively.

¹⁰The nomenclature is fixed by demanding that a choice of electric gauge group implies that all possible purely electric charges for Wilson line operators corresponding to the electric gauge group must be realized.

¹¹We denote by S_2 the Langlands duality transformation $\tau \rightarrow -\frac{1}{2\tau}$.

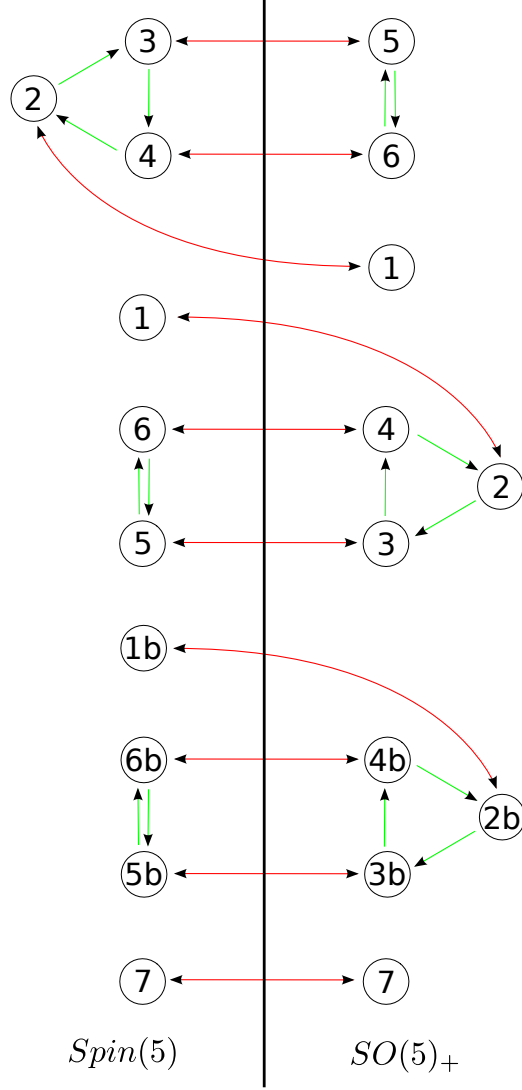


Figure 4: The diagram of the action of dualities on the massive vacua for the different B_2 theories. In red, we draw the action of Langlands S_2 -duality, and in green, T -duality (when the action is non-trivial). On the left we represent the 10 vacua of the $Spin(5)$ theory, and on the right the 10 vacua of the $SO(5)_+$ theory. The diagram of dualities for the self-dual $SO(5)_-$ theory is identical to figure 3.

2.3.1 $SO(5)_+$ vacua

Given the more limited identifications above, we obtain a longer list of extrema. The list of massive extrema for the $SO(5)_+$ theory is $(1, 2, 3, 4, 5, 6, 7, 2b, 3b, 4b)$ where the extrema $(2b, 3b, 4b)$ are obtained from $(2, 3, 4)$ by shifting by ω_1 (see figure 8 in the appendix). The doubling of the number of massive vacua arising from pure $\mathcal{N} = 1$ super Yang-Mills theory with $SO(5)_+$ gauge group is as expected from [15, 16]. We thus have ten massive vacua.

2.3.2 $SO(5)_-$ vacua

In this case, we remain with seven massive vacua. For the vacua $(2, 3, 4)$, this is as for the pure $\mathcal{N} = 1$ theory. By self- S_2 -duality, this is expected for the vacua $(1, 5, 6)$ as well.

2.3.3 $Spin(5)$ vacua

For the $Spin(5)$ theory, we again find ten massive vacua. The doubling of vacua is S_2 -dual to the duplication for $SO(5)_+$, and extrema $(1, 5, 6)$ obtain partner vacua $(1b, 5b, 6b)$ (see figure 9). The duality diagrams for the massive vacua are drawn in figure 4. The analysis in [15, 16] shows that the pure $\mathcal{N} = 1$ $Spin(5)$ theory on $\mathbb{R}^{2,1} \times S^1$ has 3 vacua, which is consistent with the one triplet under T -duality that we find on the left in figure 4. To explain the doubling of the singlet and doublet in the $Spin(5)$ theory, we refine our analysis of the unbroken gauge group further, and adapt it to include the differences between the adjoint group $SO(5)$ and the covering group $Spin(5)$. The results are in table 3.

Partition	Centralizers		Massive vacua on $\mathbb{R}^{2,1} \times S^1$		
	$SO(5)$	$Spin(5)$	$SO(5)_+$	$SO(5)_-$	$Spin(5)$
$1 + 1 + 1 + 1 + 1$	$SO(5)$	$Spin(5)$	6	3	3
$2 + 2 + 1$	$SU(2)$	$SU(2) \times \mathbb{Z}_2$	2	2	4
$3 + 1 + 1$	$\mathbb{Z}_2 \rtimes U(1)$	$\mathbb{Z}_2 \rtimes U(1)$	1	1	1
5	1	\mathbb{Z}_2	1	1	2
Total			10	7	10

Table 3: For each B_2 partition we use the Springer-Steinberg theorem to compute the centralizer inside $SO(5)$ and $Spin(5) = Sp(4)$. Finally, we compute the number of massive vacua on $\mathbb{R}^{2,1} \times S^1$ in the different theories.

For the $Spin(5)$ gauge group, we find that the centralizer for the $2 + 2 + 1$ partition and the 5 partition, contains an extra \mathbb{Z}_2 discrete factor. We can turn on a Wilson line in this \mathbb{Z}_2 group, which doubles the number of massive vacua on $\mathbb{R}^{2,1} \times S^1$ corresponding to these partitions. This matches perfectly with the doubling of the T -duality doublet and singlet extrema of the integrable system that we witness on the left of figure 4.

Summary of the Global Analysis

Thus, we have checked the duality inherited from $\mathcal{N} = 4$, including the choice of the center of the gauge group as well as the spectrum of line operators, in the case of the Lie gauge algebra $so(5)$. The $\mathcal{N} = 1^*$ theory neatly illustrates both the features of the pure $\mathcal{N} = 1$ theory as well as those of the $\mathcal{N} = 4$ theory discussed in [15, 16]. The global refinement of the analysis of all vacua can be performed for $\mathcal{N} = 1^*$ theories with *any* gauge group, but we will refrain from belaboring this particular point in the rest of our paper.

2.4 Summary and Motivation

By now, the reader may be convinced that the $\mathcal{N} = 1^*$ theory, even in the rank two case of the $so(5)$ gauge algebra, exhibits interesting elementary physical phenomena hiding in a

maze governed by modularity and ellipticity. We will isolate a subset of these interesting phenomena, and clarify the mathematical structures relevant to each. We will show that they are general, and that they can often be understood in algebraic, modular or elliptic terms. The points we will concentrate on are the following.

- We used semi-classical limits of elliptic integrable systems to render an analytic exploration of the vacuum structure coded in the low-energy effective superpotential possible. In the process, we uncovered limits of integrable systems that generalize the Inozemtsev limit [13, 21]. From the gauge theory perspective, these limits are intermediate between the confining and the Higgs regimes. In section 3 we describe these limits in more detail, and show that they are closely related to the semi-classical analysis of the $\mathcal{N} = 1^*$ theory on $\mathbb{R}^{2,1} \times S^1$ with gauge algebra \mathfrak{g} .
- We saw that a branch of massless vacua appeared as semi-classical limiting solutions, for the gauge algebra $so(5)$. The appearance of massless vacua as limiting solutions is again generic and also occurs for $su(N)$ theories, as we will show in sections 3 and 4. We will be able to analytically characterize the manifold of massless vacua for the $su(3)$ theory, including its duality properties. For the $su(4)$ theory, an analogous picture will be developed. Finally, the massless manifold of the $so(5)$ gauge theory will be scrutinized in section 6.
- We claimed that one vacuum of the $so(5)$ theory arises from turning on a \mathbb{Z}_2 Wilson line that breaks the abelian gauge group factor such as to render the vacuum massive on $\mathbb{R}^{2,1} \times S^1$. We will show that this phenomenon as well is rather generic and that we can characterize the discrete gauge group, and the Wilson line in terms of the Lie algebra data associated to the corresponding semi-classical limit. This will be demonstrated in sections 3 and 5.

The clarification of these points will occupy us for the rest of this paper. There are further open issues, some of which are enumerated in the concluding section 7.

3 Limits of Elliptic Integrable Systems and Nilpotent Orbit Theory

In this section, we firstly propose new limits of elliptic integrable systems that generalize the Inozemtsev limits performed in [13, 21]. We are motivated by the fact that these limits describe semi-classical physics of supersymmetric gauge theories in four dimensions. The existence of these limiting behaviors may also be of interest in the theory of integrable systems [13, 21, 29]. Each limit is associated to a choice of subset of the set of simple roots of (the dual of) the affine root system that enters the definition of the twisted elliptic integrable system.

In a second part of this section, we review how subsets of simple roots of affine root systems enter in the theory of nilpotent orbits. Thus, we will be able to associate semi-classical limits of the elliptic integrable system, and therefore the low-energy superpotential of $\mathcal{N} = 1^*$ theory, to a detailed description of nilpotent orbits and the component group of their centralizer. We will exploit this map in the following sections.

3.1 Semi-Classical Limits of Elliptic Integrable Systems

In this subsection, we study the (twisted) elliptic integrable systems which arise as the low-energy effective superpotentials of $\mathcal{N} = 1^*$ supersymmetric gauge theories compactified on a circle [14]. The derivation of the effective superpotential used the technique of compactification and mass deformation [10] of $\mathcal{N} = 2$ theories in four dimensions, as applied to the $su(N)$ theories in [11]. The relevant elliptic integrable systems were described in [13], where also the limits towards trigonometric and affine Toda integrable systems were presented.¹² This subsection is concerned with generalizing this analysis to include combinations of trigonometric and Toda integrable systems. These limits code possible symmetry breaking patterns of the gauge theory. The limit we take can be described as a limit towards large imaginary modular parameter τ , or as the semi-classical limit from the perspective of the $\mathcal{N} = 1^*$ gauge theory where this parameter is identified with the complex combination $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$ of the gauge coupling g and the θ angle. The procedure gives an analytical handle on the extrema of the superpotential in the semi-classical regime.

3.1.1 The Dual Affine Algebra and Non-Perturbative Contributions

The large imaginary τ expansion of the (twisted) elliptic integrable potential is known to be governed by affine algebras [14, 35–37]. Thus, it will be useful to introduce some affine algebra notation.¹³ The (untwisted) affine algebra $\hat{\mathfrak{g}} = \mathfrak{g}^{(1)}$ is built from the loop algebra of \mathfrak{g} , the central extension \hat{k} and the derivation d . We build a Cartan subalgebra of $\hat{\mathfrak{g}}$ from a Cartan subalgebra of \mathfrak{g} by adding the generators \hat{k} and d . Elements of the dual of the Cartan are denoted (λ, k, n) with the Lorentzian scalar product $(\lambda, k, n) \cdot (\lambda', k', n') = \lambda \cdot \lambda' + kn' + k'n$. If we define the imaginary root δ to be equal to $\delta = (0, 0, 1)$, the set of affine roots is

$$\hat{\Delta} = \{\alpha + m\delta | m \in \mathbb{Z} \text{ and } \alpha \in \Delta\} \cup \{m\delta | m \in \mathbb{Z} \text{ and } m \neq 0\}, \quad (3.1)$$

and the set of positive affine roots is

$$\hat{\Delta}^+ = \Delta^+ \cup \{\alpha + m\delta | m \in \mathbb{N}^* \text{ and } \alpha \in \Delta\} \cup \{m\delta | m \in \mathbb{N}^*\}. \quad (3.2)$$

A set of positive simple roots is given by adjoining the affine root $\alpha_0 = \delta - \vartheta$ (where ϑ is the highest root of \mathfrak{g}) to a simple root system of \mathfrak{g} . The theory of twisted affine algebras, their classification, their (simple, positive) roots is also pertinent here, and can be looked up in [30].

Armed with this knowledge, let's analyze how the potential behaves in the large imaginary τ limit, and how the low-energy effective superpotential codes non-perturbative corrections to gauge theory on $\mathbb{R}^{2,1} \times S^1$. The low-energy effective superpotential for the $\mathcal{N} = 1^*$ gauge theory with gauge algebra \mathfrak{g} is given by [14]

$$\mathcal{W}_{tw}(Z) = \sum_{\alpha \in \hat{\Delta}^+} g_{\nu(\alpha)} \wp_{\nu(\alpha)}(\alpha \cdot Z; \tau) \quad (3.3)$$

¹²This analysis extended the one performed in [21]. See also [29].

¹³See e.g. [30] for the theory of affine Kac-Moody algebras.

where the index $\nu(\alpha)$ is defined by

$$\nu(\alpha) = \frac{|\alpha_{long}|^2}{|\alpha|^2} = \frac{2}{|\alpha|^2} \quad (3.4)$$

and the short and long root coupling constants are expressed in terms of a single constant g by

$$g_\nu = \frac{g}{\nu}. \quad (3.5)$$

We normalize the long roots to have length squared two. The gauge coupling τ is given by the ratio of the periods of the torus $\tau = \omega_2/\omega_1$. To perform the semi-classical large imaginary τ expansion, we can exploit the result

$$\wp(2\omega_1 x; \omega_1, \omega_2) = -\frac{\pi^2}{12\omega_1^2} E_2(q) + \frac{\pi^2}{4\omega_1^2} \csc^2(\pi x) - \frac{2\pi^2}{\omega_1^2} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \cos 2\pi n x, \quad (3.6)$$

where $q = e^{2\pi i \tau}$. This expansion is valid whenever the series is convergent, which requires $|q| < 1$ or equivalently that $\tau \in \mathcal{H}$, and also $|\Im(x)| < \Im(\tau)$. The space \mathcal{H} is the upper-half plane of complex numbers with positive imaginary part. For $x \in \mathcal{H}$ or $x \in \mathbb{R} \setminus \mathbb{Z}$ we can use the further expansion:

$$-4 \sum_{k=1}^{\infty} k e^{2\pi i k x} = \csc^2 \pi x, \quad (3.7)$$

to find

$$\wp(2\omega_1 x; \omega_1, \omega_2) = -\frac{\pi^2}{12\omega_1^2} E_2(q) - \frac{\pi^2}{\omega_1^2} \sum_{n=1}^{\infty} n \left[e^{2\pi i n x} + \sum_{m=1}^{\infty} q^{nm} (e^{-2\pi i n x} + e^{2\pi i n x}) \right] \quad (3.8)$$

For the twisted Weierstrass function \wp_ν defined for $\nu \in \mathbb{N}^*$ by

$$\wp_\nu(z; \omega_1, \omega_2) = \sum_{k=0}^{\nu-1} \wp \left(z + \frac{k}{\nu} 2\omega_1; \omega_1, \omega_2 \right), \quad (3.9)$$

we have the counterpart

$$\begin{aligned} \wp_\nu(2\omega_1 x; \omega_1, \omega_2) &= -\frac{\nu\pi^2}{12\omega_1^2} E_2(q) + \frac{\nu^2\pi^2}{4\omega_1^2} \csc^2(\pi\nu x) - \frac{2\nu^2\pi^2}{\omega_1^2} \sum_{n=1}^{\infty} \frac{nq^{n\nu}}{1-q^{n\nu}} \cos 2\pi n\nu x \\ &= -\frac{\nu\pi^2}{12\omega_1^2} E_2(q) - \frac{\nu^2\pi^2}{\omega_1^2} \sum_{n=1}^{\infty} n \left[e^{2\pi i n\nu x} + \sum_{m=1}^{\infty} q^{nm\nu} (e^{-2\pi i n\nu x} + e^{2\pi i n\nu x}) \right] \end{aligned} \quad (3.10)$$

Again, this expansion is valid for $x \in \mathbb{C} \setminus \mathbb{Z}$ such that $0 \leq \Im(x) < \Im(\tau)$. It should be clear that the part of the argument of the Weierstrass function proportional to τ plays a crucial role in the Taylor series in the large τ limit. This is illustrated by the fact that for any $0 < a < 1$ and any $b \in \mathbb{R}$,

$$\lim_{\tau \rightarrow i\infty} \wp(a\tau + b; \tau) = -\frac{\pi^2}{3}. \quad (3.11)$$

It is therefore useful to separate the argument into a part proportional to τ and a part that will not grow with τ , by setting

$$Z = X + \tau Y, \quad (3.12)$$

where X and Y are complex variables. At this stage, this decomposition is arbitrary. We have doubled the number of degrees of freedom, and we will use this redundancy in subsection 3.1.2 to impose the value of Y . Plugging this parametrization into the (twisted) Weierstrass function yields

$$\wp_\nu(2\omega_1\alpha \cdot Z; \omega_1, \omega_2) = -\frac{\nu\pi^2}{12\omega_1^2}E_2(q) - \frac{\nu^2\pi^2}{\omega_1^2} \sum_{n=1}^{\infty} n \left[q^{n\nu\alpha \cdot Y} e^{2in\pi\nu\alpha \cdot X} \right] \quad (3.13)$$

$$+ \sum_{m=1}^{\infty} q^{nm\nu} \left(q^{-n\nu\alpha \cdot Y} e^{-2in\pi\nu\alpha \cdot X} + q^{n\nu\alpha \cdot Y} e^{2in\pi\nu\alpha \cdot X} \right) \Big]. \quad (3.14)$$

Using these expansion formulas for the potential, we arrive at a sum of exponential terms, each associated to a positive affine root

$$\mathcal{W}_{tw}(\hat{Z}) = -\frac{g\pi^2}{\omega_1^2} \left(\frac{|\Delta^+|}{12} E_2(q) + \sum_{n=1}^{\infty} n \left[\sum_{\hat{\alpha} \in \hat{\Delta}^+} \nu(\hat{\alpha}) q^{n\nu(\hat{\alpha})\hat{\alpha} \cdot \hat{Y}} e^{2\pi i n \nu(\hat{\alpha})\hat{\alpha} \cdot X} \right] \right). \quad (3.15)$$

We have used the notations $\hat{X} = (X, 0, 0) = X$, $\hat{Y} = (Y, 1, 0)$ and $\hat{Z} = \hat{X} + \tau\hat{Y}$ so that for any affine root $\hat{\alpha} = \alpha + m\delta \in \hat{\Delta}$, we have the equality $\hat{\alpha} \cdot \hat{Y} = \alpha \cdot Y + m$. We also define ν on affine roots with non-zero real part by $\nu(\hat{\alpha}) = \nu(\alpha + m\delta) = \nu(\alpha)$, so that

$$\hat{\alpha}^\vee = \frac{2}{|\hat{\alpha}|^2} \hat{\alpha} = \frac{2}{|\alpha|^2} \hat{\alpha} = \nu(\hat{\alpha}) \hat{\alpha}. \quad (3.16)$$

We have arbitrarily declared $\nu(m\delta) = 0$.¹⁴ The form of the exponents in equation (3.15) suggests switching from the affine root system to its dual

$$\mathcal{W}_{tw}(\hat{Z}) = -\frac{g\pi^2}{\omega_1^2} \left(\frac{|\Delta^+|}{12} E_2(q) + \sum_{n=1}^{\infty} n \left[\sum_{\hat{\alpha}^\vee \in (\hat{\Delta}^+)^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} q^{n\hat{\alpha}^\vee \cdot \hat{Y}} e^{2\pi i n \hat{\alpha}^\vee \cdot X} \right] \right). \quad (3.17)$$

In the sum, we again disregard the terms associated to purely imaginary roots.

In the gauge theory, the semi-classical expression (3.17) has an interpretation as a sum over three-dimensional monopole-instanton contributions [14].¹⁵ Note that the purely four-dimensional instanton terms associated to the imaginary roots contribute a τ dependent, but position independent term in the superpotential. We have two forms for the final expression. One expression (namely (3.15)) is in terms of the root system we started out with, the other (namely (3.17)) in terms of co-roots. Both forms are equally canonical,

¹⁴We note that low-energy effective superpotential is ambiguous up to a purely q -dependent term.

¹⁵See [36] for a graphical representation of the non-perturbative states that contribute, in terms of D-brane systems in string theory.

due to the fact that both the electric Wilson line variable and the dual photon variable are present in the potential and are interchanged under Langlands duality. This is a manifestation of the S-duality of the parent $\mathcal{N} = 4$ theory. In a given semi-classical expansion (i.e. $\tau \rightarrow i\infty$), we may more easily read expression (3.17), which has an interpretation as a sum over magnetic monopole instantons in this limit.

3.1.2 Semi-Classical Limits

Concretely, we take semi-classical limits as follows. We consider a particular isolated extremum whose positions Z depend only on τ (up to discrete equivalences that depend on the gauge group). We assume that at weak coupling, the limit

$$Y \equiv \lim_{\tau \rightarrow i\infty} \frac{1}{\tau} Z(\tau) \quad (3.18)$$

exists and we define $X(\tau) = Z(\tau) - \tau Y$. Note that for any $\tau \in \mathcal{H}$ we have $Z(\tau) = X(\tau) + \tau Y$ as before, and the parametrization Y is a vector that is independent of τ and which characterizes the extremum (or several extrema) under consideration. It is a non-trivial task to enumerate the set of vectors Y that give rise to isolated extrema. We will also deal with continuous branches of extrema, for which the definition (3.18) has no intrinsic meaning. In this case we can nevertheless choose an arbitrary set of coordinates of the branch, and take the limit while keeping these coordinates fixed. Depending on the choice of parametrization, this may lead to a continuous set of values for the vector Y . From now on, when studying a given extremum, we trade the variable $Z(\tau)$ for the variable $X(\tau)$ which is finite in the limit we want to perform, and use the expansion (3.17).

Before doing so, let's choose a basis of simple roots $(\alpha_1, \dots, \alpha_r)$ in the root system Δ . Then $(\alpha_0, \alpha_1, \dots, \alpha_r)$ are the simple roots of the affine root system $\hat{\Delta}$. The dual root system has a set of simple roots $((\alpha_0)^\vee, \alpha_1^\vee, \dots, \alpha_r^\vee)$. To be more explicit about the semi-classical limit, we must distinguish between variables that sit on the boundary of the fundamental alcove, and those that reside inside. We therefore choose a vector Y in the fundamental affine Weyl chamber (or fundamental alcove), which implies that $(\alpha_i)^\vee \cdot \hat{Y} \geq 0$ for $i = 0, 1, \dots, r$. We decompose the positive roots in terms of simple roots of the dual of the affine algebra, and the vector Y in the weight space in terms of affine fundamental weights $\hat{\pi}_i$:

$$\hat{\alpha}^\vee = \sum_{i=0}^r n_i \alpha_i^\vee \quad \hat{Y} = \sum_{i=0}^r Y_i \hat{\pi}_i \quad (3.19)$$

where the n_i are non-negative integers,

$$\hat{\pi}_i = (\pi_i; a_i^\vee; 0) \quad \hat{\pi}_0 = (0; 1; 0), \quad (3.20)$$

and a_i^\vee denote the co-marks of the Lie algebra. The fundamental weights satisfy the orthonormality conditions $(\hat{\pi}_i, \alpha_j^\vee) = \delta_{ij}$, so that $Y_i = \alpha_i^\vee \cdot \hat{Y} \geq 0$ and

$$\hat{\alpha}^\vee \cdot \hat{Y} = \sum_{i=0}^r n_i Y_i. \quad (3.21)$$

Note that the definition of $\hat{Y} = (Y, 1, 0)$ gives a linear relation between the $r+1$ coordinates Y_i ,

$$1 = Y_0 + \sum_{i=1}^r Y_i a_i^\vee = \sum_{i=0}^r Y_i a_i^\vee. \quad (3.22)$$

Similarly we define $X_i = \alpha_i^\vee \cdot X$, and have the constraint $\sum_{i=0}^r X_i a_i^\vee = 0$. The distinction we now make is between those variables Y_i that lie on the boundary of the fundamental alcove, and those that lie inside. This will fix the leading behavior of the extrema that we focus on. For $Y_i = 0$, we note that there is an infinite set of non-perturbative contributions that needs to be taken into account in the semi-classical limit, and in particular, we need to resum them to the trigonometric term (as in equation (3.7)). The set of roots α^\vee for which this phenomenon occurs will again form a root system. Thus, to leading order in the modular parameter $q = e^{2\pi i \tau}$, we will have a trigonometric integrable system corresponding to a choice of subset of simple roots inside the affine simple root system. In a second step, by assumption, we have the remaining coordinates Y_j that do not vanish to leading order in τ . As a consequence (of formula (3.17)), these directions Y_j lead to subleading exponential terms.

More in detail, let's group positive roots by their inner products with \hat{Y} and form the sets:

$$(\hat{\Delta}_t^+(Y))^\vee = \left\{ \hat{\alpha}^\vee \in (\hat{\Delta}^+)^\vee \mid \hat{\alpha}^\vee \cdot \hat{Y} = 2\omega_1 t \right\}, \quad (3.23)$$

and also the spectrum S of such inner products

$$S(Y) = \left\{ t \in \mathbb{R} \mid (\hat{\Delta}_t^+(Y))^\vee \neq \emptyset \right\}. \quad (3.24)$$

The spectrum of inner products without zero will be denoted $S(Y)^* = S(Y) - \{0\}$. The set of roots with zero inner product is finite while the full spectrum $S(Y)$ is generically infinite, due to the infinite nature of the affine root system. The superpotential

$$\mathcal{W}_{tw}(\hat{Z}) = -\frac{g\pi^2}{\omega_1^2} \left(C(\tau) + \sum_{n=1}^{\infty} n \left[\sum_{t \in S(Y)} q^{nt} \sum_{\hat{\alpha}^\vee \in (\hat{\Delta}_t^+)^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} e^{2\pi i n \hat{\alpha}^\vee \cdot X} \right] \right), \quad (3.25)$$

will split into two sets of terms. Note that the exponents of q are non-negative, so that the expression remains finite when we take the limit $q \rightarrow 0$. As mentioned previously, the first split happens between terms with zero inner product and non-zero inner product:

$$\mathcal{W}_{tw}(\hat{Z}) = -\frac{g\pi^2}{\omega_1^2} \left(C(\tau) - \frac{1}{4} \sum_{\hat{\alpha}^\vee \in (\hat{\Delta}_0^+)^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} \csc^2(\pi \hat{\alpha}^\vee \cdot X) \right. \quad (3.26)$$

$$\left. + \sum_{t \in S(Y)^*} \sum_{n=1}^{\infty} n q^{nt} \left[\sum_{\hat{\alpha}^\vee \in (\hat{\Delta}_t^+)^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} e^{2\pi i n \hat{\alpha}^\vee \cdot X} \right] \right). \quad (3.27)$$

We obtain a sum of a trigonometric and an exponential system

$$\mathcal{W}_{tw}(\hat{Z}) = -\frac{g\pi^2}{\omega_1^2} \left(C(\tau) - \frac{1}{4} \mathcal{W}_{trig}^Y(X) + \sum_{t \in S(Y)^*} \sum_{n=1}^{\infty} n q^{nt} \mathcal{W}_{exp}^{(n,t,Y)}(X) \right), \quad (3.28)$$

where

$$\mathcal{W}_{trig}^Y(X) = \sum_{\hat{\alpha}^\vee \in (\hat{\Delta}_0^+(Y))^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} \csc^2(\pi \hat{\alpha}^\vee \cdot X) \quad (3.29)$$

$$\mathcal{W}_{exp}^{(n,t,Y)}(X) = \sum_{\hat{\alpha}^\vee \in (\hat{\Delta}_t^+(Y))^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} e^{2\pi i n \hat{\alpha}^\vee \cdot X}. \quad (3.30)$$

The behavior of the subdominant system is intricate. A first stab at the subdominant system consists in realizing that the remaining variables (indexed by the set $\bar{J}_0 = \{0, 1, \dots, r\} \setminus J_0$ where J_0 is the set of coordinates with zero inner product) will all have a leading exponential term. These exponentials, combined with the constraint equation (3.22), may give rise to exponential interactions, stabilized by an exponential interaction of opposite sign. The affine Toda potential is an example of this type of subdominant potential. Roughly speaking, this reasoning goes through, but the devil is in the details. The first complicating factor is the influence of the dominant terms on the subdominant terms when searching for an equilibrium position. In particular, corrections to equilibrium positions for leading coordinates may strongly influence subdominant contributions. Particular equilibrium configurations for the leading trigonometric system can also give rise to subtle and persistent cancellations in the coefficients of subdominant exponential terms. There may also be a staircase of subdominant terms, each with its own limiting behavior. Even a continuous set of limiting behaviors can occur. Moreover, the solutions to the trigonometric system are only known as zeroes of orthogonal polynomials, making this process hard to carry through analytically in full generality.

Therefore, we develop only a partial picture of the integrable systems that result in the limit. Still, we provide a generalization of the limit discussed in [13] in the following subsection, and useful heuristics based on the examples in sections 2, 4 and 5.

3.1.3 The Trigonometric, Affine Toda and Intermediate Limits

Here we will treat the special case in which no cancellation of (sub)leading exponentials occurs, and in which the subleading exponential integrable system stabilizes all the remaining coordinates and leads to an isolated extremum. We can then analytically solve for the remaining variables. Due to the constraint equation we have that the set $J_0 \subsetneq \{0, 1, \dots, r\}$ is a true subset of the set of simple roots (we identify a simple root α_i with its index i). We obtain a trigonometric integrable system for the root system corresponding to the simple roots in J_0 . This system gives solutions for $|J_0|$ of the $r + 1$ variables X_i . Let t_1 be the smallest non-zero element of the spectrum $S(Y)$. At the next level in the q -expansion, we find contributions corresponding to the set $(\hat{\Delta}_{t_1}^+(Y))^\vee$, which is equal to a set of positive roots.

The final Toda integrable system is a sum over $|\bar{J}_0|$ vectors where \bar{J}_0 is the complement of the set of affine simple roots that enter the trigonometric system, by assumption. We then obtain

$$\mathcal{W}_{Toda}^{(n=1,t_1,Y)}(X) = C \sum_{i \in \bar{J}_0} \frac{1}{\nu(\alpha_i^\vee)} e^{2\pi i \alpha_i^\vee \cdot X}.$$

In the last equality we have indicated the fact that for each individual index i , there may be a renormalization of the constant $\tilde{\nu}$ in front of the exponential term, due to various roots contributing to the same exponential behavior. The constraint equation then gives

$$1 = \sum_{i \in \bar{J}_0} Y_i a_i^\vee = 2\omega_1 t_1 \sum_{i \in \bar{J}_0} a_i^\vee \quad (3.31)$$

from which we extract t_1 and finally

$$\hat{Y} = \frac{\sum_{i \in \bar{J}_0} \hat{\omega}_i}{\sum_{i \in \bar{J}_0} a_i^\vee}. \quad (3.32)$$

After projection on the finite part, we find:

$$Y = \frac{\sum_{i \in \bar{J}_0} \omega_i}{\sum_{i \in \bar{J}_0} a_i^\vee}. \quad (3.33)$$

where we define $\omega_0 = 0$. Here the dependence on t_1 has disappeared. We can simply use $Y_{\bar{J}_0}$ as an ansatz, for every non-empty set $\bar{J}_0 \subset \{0, 1, \dots, r\}$, where we have defined

$$Y_J = \frac{\sum_{i \in J} \omega_i}{\sum_{i \in J} a_i^\vee}. \quad (3.34)$$

On the condition that the subleading exponentials have non-vanishing coefficients, this gives the semi-classical (linear order in τ) values for the Y coordinates of the integrable system. Namely, a first set sits at an extremum of the trigonometric integrable system, and a second set at the extrema of the affine Toda system. Known applications of this ansatz are the following. A first extreme case is $\bar{J}_0 = \emptyset$ and $Y = 0$. Then $\Delta_0^+(Y) = \Delta^+$ and we recover the trigonometric potential only. The other extreme case is $\bar{J}_0 = \{0, 1, \dots, r\}$ and $Y = \rho/h^\vee$ (where ρ is the Weyl vector and h^\vee the dual Coxeter number of the gauge algebra). We then obtain the affine Toda potential for the algebra $(\mathfrak{g}^{(1)})^\vee$, as described in [13]. There are many intermediate cases that follow the above pattern, or an even more intricate one.¹⁶ Examples are provided in sections 4 and 5, and we already saw some in section 2. It would be desirable to have a full classification of semi-classical limits. The $\mathcal{N} = 1^*$ gauge theory provides intuition in the case of the (twisted) elliptic Calogero-Moser system with particular coupling constants – the question in the integrable system context is even more general.

3.2 The Nilpotent Orbit Theory of Bala-Carter and Sommers

From the previous subsection, we conclude that we can associate semi-classical extrema of the elliptic integrable system to subsets of the (dual) affine simple root system. In this

¹⁶In the gauge theory, these cases correspond respectively to a fully Higgsed vacuum, confining pure $\mathcal{N} = 1$ dynamics, and partial Higgsing.

section, we show that there is another way to understand the relevance of these subsets, in terms of nilpotent orbit theory and the physics of $\mathcal{N} = 1^*$ theory on $\mathbb{R}^{2,1} \times S^1$.

Firstly, let us briefly review highlights of nilpotent orbit theory. See e.g. the textbooks [22–25] for a gentler introduction. The Bala-Carter classification of nilpotent orbits of simple algebraic groups goes as follows. Each nilpotent orbit of a Lie algebra \mathfrak{g} of a connected, simple algebraic group G is a distinguished nilpotent orbit of a Levi subalgebra [18, 19]. Levi subalgebras of \mathfrak{g} correspond to subsets of simple roots of \mathfrak{g} (up to conjugation by the Weyl group). Distinguished nilpotent orbits are those for which the nilpotent element does not commute with a non-central semi-simple element.¹⁷

Furthermore, there is generalization of the Bala-Carter correspondence by Sommers [20]. There is a one-to-one correspondence between nilpotent elements n and conjugacy classes of the component group $Comp(n)$ of the centralizer on the one hand, and pairs (\mathfrak{l}, n) of pseudo-Levi subalgebras \mathfrak{l} and distinguished nilpotent elements n in \mathfrak{l} on the other hand. The correspondence is up to group conjugacy. A pseudo-Levi subalgebra corresponds, by definition, to a subset of the simple root system of \mathfrak{g} extended by the lowest root $-\theta$. This classification allows for the unified calculation of all the component groups of nilpotent orbits of simple Lie algebras [20]. The relevance of these results can be gleaned from the $so(5)$ example discussed in section 2, from the semi-classical limits analyzed above, and can also generically be argued for, as follows.

3.3 The Bridge between Gauge Theory and Integrable System

Semi-classical solutions to the F-term equations of motion for $\mathcal{N} = 1^*$ theory on \mathbb{R}^4 are classified by matching them onto nilpotent orbits [9]. When we compactify the gauge theory on S^1 , there are further aspects of nilpotent orbits that come into play. In particular, we will allow for Wilson lines in the unbroken gauge group. If the latter contains topologically non-trivial conjugacy classes, i.e. conjugacy classes in the component group of the centralizer, then we need to consider each of these configurations separately.

As we saw, there is a one-to-one correspondence between the pair (nilpotent orbit, conjugacy class of component group) and pseudo-Levi subalgebras. The trivial conjugacy classes will correspond to a collection of non-affine simple roots. Each inequivalent choice of subset that necessarily includes the affine root will correspond to a non-trivial conjugacy class of a component group. These are classified by Bala-Carter-Sommers theory, which therefore is crucial in classifying semi-classical configurations for $\mathcal{N} = 1^*$ theory compactified on a circle. The example of the $\mathcal{N} = 1^*$ gauge theory with G_2 gauge group discussed in section 5 will neatly illustrate our reasoning.

Before we turn to this application, we demonstrate the use of the semi-classical limit in example systems. In particular, the techniques developed in this section allow for the analysis of the physics and duality properties of the massless vacua of $su(N)$ theories.

¹⁷ We illustrate the application of these concepts more concretely in the case of B_2 , A_{N-1} and G_2 in sections 2, 4 and 5, and the mathematics literature contains much more detail.

4 The $\mathcal{N} = 1^*$ Theory with Gauge Algebra A_{N-1}

In this section, we concentrate on the $\mathcal{N} = 1^*$ theory with $su(N)$ gauge algebra. This theory has many simplifying features. In particular, the unbroken gauge group in all semi-classical vacua of the $SU(N)/\mathbb{Z}_N$ theory is connected, so that the component group (in the adjoint group) is trivial.¹⁸ Indeed, from the mathematical point of view we have that Bala-Carter theory coincides with Bala-Carter-Sommers theory. In the A -type case, all pseudo-Levi subalgebras are equivalent to Levi subalgebras, since the lowest root is Weyl equivalent to any other simple root. Thus, in this theory, we can isolate new semi-classical limits of the integrable system, and the corresponding gauge theory physics from other interesting features of $\mathcal{N} = 1^*$ theories when compactified on S^1 . We will find branches of massless vacua for low rank, characterize their equilibrium positions, analyze their superpotential and study how these vacua behave under duality.¹⁹

To understand the fate of semi-classically massless and massive vacua in $su(N)$ $\mathcal{N} = 1^*$ theory, we again take the elliptic Calogero-Moser Hamiltonian as our starting point [11]. For starters, we analyze this effective superpotential in the semi-classical regime $\tau \rightarrow i\infty$ and classify extrema of the integrable system using the technique laid out in section 3. We will be able to promote parts of our limiting knowledge to exact statements at finite coupling.

4.1 Semi-Classical Preliminaries

As argued previously, a classification of extrema is governed by pseudo-Levi subalgebras, in turn determined by Weyl (i.e. permutation) inequivalent subsets of the affine root system (whose Dynkin diagram is a circle with N nodes). The number of inequivalent subsets of roots (except all of them) is the number of partitions of N . In more detail, we let $\Delta = \{\alpha_1, \dots, \alpha_{N-1}\}$ be a set of simple roots of A_{N-1} . For any subset $\Theta \subset \Delta$ we construct a partition of N . We can write Θ uniquely as a disjoint union of sets of the form $\Delta_{k_i, d_i} = \{\alpha_{k_i}, \dots, \alpha_{k_i+d_i-2}\}$ where $d_i \geq 2$. If our choice of subset Θ is

$$\Theta = \bigcup_i \Delta_{k_i, d_i} \quad (4.1)$$

then the partition is $1 + \dots + 1 + \sum_i d_i = N$ with as many 1's as necessary to obtain a partition of N .

For each choice of subsystem, we know the corresponding centralizer subgroup in the complexification of $SU(N)$. We denote the latter by $GL(N)$, the group of size N invertible matrices with complex entries. The algebra of the centralizer is given in [23]. With the notation $r_i = |\{j | d_j = i\}|$ for the number of times a representation of dimension i occurs in the $sl(2)$ representation spanned by the adjoint scalars, so that

$$\sum_i i r_i = N, \quad (4.2)$$

¹⁸The center of the gauge group would play the leading role in the discussion of the global aspects of the gauge theory. See [15, 16], and our subsection 2.3.

¹⁹A preliminary discussion of massless vacua can be found in [31].

the centralizer algebra is

$$\left(\prod_i A_{r_i-1} \right) \times u(1)^k, \quad (4.3)$$

where $k = |\{i | r_i > 0\}| - 1$ (i.e. the number of distinct dimensions minus one). Then the global structure of the centralizer group is [22]

$$S \left(\prod GL(r_i)_\Delta^i \right), \quad (4.4)$$

where the Δ denotes the diagonal copy of $GL(r_i)$ inside $GL(r_i)^i$, and the S in front means that we keep only the matrices with unit determinant. This is the centralizer group in the complexification of $SU(N)$. The counting of the abelian factors in this group goes as follows: there is one abelian factor for each term in the product and the constraint of unit determinant reduces the total number of abelian factors by one. In terms of pseudo-Levi subalgebras, a group with no abelian factor is obtained from a set of roots Θ containing all the simple roots except d of them (where d divides N) equally spaced on the cyclic affine Dynkin diagram. In other words, one takes disconnected groups of $d - 1$ roots on the affine Dynkin diagram, where d is a divisor. These give rise to the massive vacua of the $\mathcal{N} = 1^*$ theory that were described in [6, 11]. The corresponding semi-classical limits of the integrable system are well-understood. We wish to advance the more general case. We will do this on a case-by-case basis, working our way up in rank.

In the next subsections, we use the semi-classical limiting technique to gain information on the massless vacua of the first non-trivial low rank cases. For the $su(3)$ theory, we will complete the picture at finite coupling, while for the $su(4)$ algebra, we present a few features that will be typical of higher rank.

4.2 The Gauge Algebra $su(3)$, the Massless Branch and the Singularity

We remind the reader that the superpotential for the $su(3)$ algebra [11] can be parametrized in terms of the coordinates z_i with $i = 1, 2, 3$ where we can use a shift symmetry to put $z_3 = 0$:

$$\begin{aligned} \mathcal{W}_{A_2} &= \wp(z_1 - z_2) + \wp(z_2 - z_3) + \wp(z_3 - z_1) \\ &= \wp(z_1 - z_2) + \wp(z_2) + \wp(z_1) \\ &= \wp(Z_1) + \wp(Z_2) + \wp(Z_1 + Z_2). \end{aligned} \quad (4.5)$$

In the last line, we have used the more intrinsic parametrization in terms of the coordinates Z_i associated to the fundamental weights.

Semi-classical analysis

When we apply our program of identifying vacua in the semi-classical limit to the case of $su(3)$, we recuperate the known results for the massive vacua, and find new results for massless vacua.

- For the choice $J_0 = \emptyset$, which corresponds to the partition $1+1+1$, we set the leading behavior $Y_0 = Y_1 = Y_2 = \frac{1}{3}$. One finds three confined massive vacua (with $k = 0, 1, 2$) at

$$(z_1, z_2, z_3) = \left(\frac{k}{3} + \frac{2}{3}\tau, \frac{2k}{3} + \frac{1}{3}\tau, 0 \right). \quad (4.6)$$

These extremal positions are exact and the superpotential in these vacua is known [11].

- For the pick $J_0 = \{1\}$, namely the partition $3 = 1+2$, we choose $Y_0 = Y_2 = \frac{1}{2}$, and at first order the A_1 trigonometric system fixes $X_1 = \frac{1}{2}$. We analyze the potential near this equilibrium by expanding $\mathcal{W}_{A_2}(\frac{1}{2} + \delta X_1, \frac{\tau}{2} + X_2)$ in perturbation theory in δX_1 , and as a function of X_2 . We find that the first coordinate is corrected as follows

$$\delta X_1 = -\frac{4i(e^{2i\pi X_2} - e^{-2i\pi X_2})\sqrt{q}}{\pi} - \frac{16i(e^{4i\pi X_2} - e^{-4i\pi X_2})q}{\pi} + \dots \quad (4.7)$$

Plugging this correction into the superpotential leads to a superpotential which to the relevant order no longer depends on X_2 , and in fact, is equal to zero. We have checked this to order q^4 . These facts point towards the existence of a branch of massless vacua, with zero superpotential along the whole branch. We will obtain full analytic control of this branch below.

- Finally, for the choice $J_0 = \{1, 2\}$, namely the partition 3, one obtains the A_2 trigonometric potential. This potential has a real extremum, the fully Higgsed vacuum

$$(z_1, z_2, z_3) = \left(\frac{2}{3}, \frac{1}{3}, 0 \right), \quad (4.8)$$

as well as complex massless extrema which form a portion of the same branch of vacua with zero superpotential just mentioned.

The Massless Branch and the Singularity

Semi-classically, we have found evidence for the existence of a massless branch of vacua with zero superpotential. In the following, we will concentrate on describing the properties of this branch analytically, at any finite coupling τ . Together with the known results about massive vacua that our analysis also recovers, we thus obtain all the vacua of the $\mathcal{N} = 1^*$ theory with $su(3)$ gauge algebra exactly.

Firstly, we introduce some notation. We will denote the elliptic curve variables as

$$\begin{aligned} \mathcal{X}_i &= \wp(Z_i) \\ \mathcal{Y}_i &= \wp'(Z_i) \end{aligned} \quad (4.9)$$

for $i = 1, 2, 3$, where $Z_3 = -Z_1 - Z_2$ by convention. The points $(\mathcal{X}_i, \mathcal{Y}_i)$ all lie on the same elliptic curve, parametrized by τ , and described by the equation

$$\mathcal{Y}^2 = 4\mathcal{X}^3 - g_2\mathcal{X} - g_3. \quad (4.10)$$

The equations for extremality of the superpotential then read

$$\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Y}_3. \quad (4.11)$$

Moreover, the addition theorem for the elliptic Weierstrass function implies

$$\mathcal{X}_i = \frac{1}{4} \left(\frac{\mathcal{Y}_j - \mathcal{Y}_k}{\mathcal{X}_j - \mathcal{X}_k} \right)^2 - \mathcal{X}_j - \mathcal{X}_k, \quad (4.12)$$

where i, j, k take three distinct values in the set $\{1, 2, 3\}$. Thus, we see that there are two possibilities: either the superpotential is zero

$$\mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 = 0, \quad (4.13)$$

or we must have that

$$\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}_3. \quad (4.14)$$

We split the analysis of the extrema according to these two cases. Firstly, we consider the case in which we have the equality (4.14). This equation, together with extremality shows that $Z_1 \equiv Z_2 \equiv Z_3$ modulo a period. This implies that all Z_i equal a non-trivial third of a period of the torus, and gives rise to 4 inequivalent vacuum solutions, which are the known massive vacua [11]. The superpotential is three times the Weierstrass function evaluated at a third period.

Let us return then to the first possibility, which is that the superpotential is zero, equation (4.13). By eliminating the variables \mathcal{Y}_i through the curve equation and extremality, we obtain two equations characterizing the massless branch

$$\begin{aligned} \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 &= 0 \\ \mathcal{X}_1^2 + \mathcal{X}_2^2 + \mathcal{X}_3^2 &= \frac{g_2}{2}. \end{aligned} \quad (4.15)$$

These equations are gauge invariant. Solving for the variables \mathcal{Y}_i will provide a further double cover of this space. Moreover, we mod out the space by the discrete gauge symmetry S_3 , which exchanges the three indices $\{1, 2, 3\}$ of the variables \mathcal{X}_i (and flips the sign of the variables \mathcal{Y}_i if the permutation is odd, exchanging the two sheets of the cover). We can parametrize the curve more explicitly by eliminating more variables. A description of the curve in terms of two variables is

$$\mathcal{X}_1^2 + \mathcal{X}_1 \mathcal{X}_2 + \mathcal{X}_2^2 = \frac{g_2}{4}. \quad (4.16)$$

This equation parametrizes a complex line.

Note that at the values τ_0 of the complexified gauge coupling where the fourth Eisenstein series g_2 is zero, the complex line has a singular point at $\mathcal{X}_1 = \mathcal{X}_2 = 0$. The singularity is a crucial feature of the massless branch. The zeros of g_2 in the τ upper half-plane are exactly the $SL(2, \mathbb{Z})$ images of $\tau_0 = e^{2\pi i/3}$, which is the only zero of g_2 in the fundamental

domain.²⁰ Thus, at these couplings the massless branch develops a singularity. These are elliptic points of order three.

Finally, we note that the conditions that all \mathcal{X}_i be equal (which is valid for the 4 vacua associated to third periods), and that the superpotential vanish can both be satisfied at the singular points. More precisely, for each given singular coupling in the $SL(2, \mathbb{Z})$ orbit of τ_0 , one of the four formerly massive vacua becomes massless and joins the massless branch. The fact that a massive vacuum becomes massless at this coupling may indicate a higher order critical point, and the existence of an interacting $\mathcal{N} = 1$ superconformal field theory. The value of the critical coupling points towards a natural candidate for this theory, which is the Argyres-Douglas theory [32] broken to $\mathcal{N} = 1$ [33].

In fact, the analysis of $\mathcal{N} = 2$ $SU(3)$ theory with an adjoint hypermultiplet reveals that the Seiberg-Witten curve has eight cusps [6]. When we analyze the cusps at values of the moduli such that they coincide with vacua that would be massive at generic coupling, we find that the number of cusps reduces to four.²¹ Of these four cusps, one is associated to a pure $SU(3)$ theory, and the other three correspond to a $SU(2)$ theory with a massless fundamental hypermultiplet at a (generalized) Argyres-Douglas point [34]. Since the $\mathcal{N} = 1^*$ massive vacua are invariant under $\Gamma_0(3)$, we can classify the singular couplings τ into $\Gamma_0(3)$ cosets of $SL(2, \mathbb{Z})$ according to which massive vacuum becomes massless at the given singular coupling. We find that at $\tau = \frac{-1}{\tau_0+2}$ the Higgs vacuum becomes massless, while at $\tau = \tau_0 + 2$ the confined vacuum situated on the imaginary axis becomes massless, at $\tau = \tau_0 + 1 = e^{\pi i/3}$, its T-dual and at $\tau_0 = e^{2\pi i/3}$ the third confined vacuum. Thus, the $SL(2, \mathbb{Z})$ action on these Argyres-Douglas singularities coincides with the action of the duality group on the four massive vacua of the $\mathcal{N} = 1^*$ theory. From the action of the T-transformation, we can identify the confined vacua with the $SU(2)$ cusps and the Higgs vacuum with the pure $SU(3)$ cusp [6]. Our analysis provides a concrete picture for how the transformation properties of the massive phases are locked with the duality properties of the cusps.

At generic coupling τ , the duality properties of the massive vacua are well-known. We find that the massless branch, in the description in terms of elliptic curve variables, is invariant under the action of the T-transformation, since the fourth Eisenstein series is. Moreover, under the S-transformation, the variables \mathcal{X}_i transform with weight two, as one expects from their definition in terms of the elliptic Weierstrass function. Thus, the branch is self-dual under the full modular group (or more precisely, is mapped to an equivalent, scaled branch at dual coupling).

The Massless Branch in the Toroidal Variables

The description of the massless branch was straightforward in terms of gauge invariant polynomials of the variables \mathcal{X}_i . Still, we can ask for the description of the massless branch

²⁰It is easy to show that $\tau_0 = e^{2\pi i/3}$ is a zero of $E_4 = \frac{3}{4\pi^4}g_2$ using $E_4(\tau_0) = E_4(\tau_0 + 1) = E_4(-1/\tau_0) = \tau_0^4 E_4(\tau_0) = \tau_0 E_4(\tau_0)$. There remains to show that there is no other zero. We use the formula $\text{ord}_{i\infty} + \frac{1}{2}\text{ord}_i + \frac{1}{3}\text{ord}_{\tau_0} + \sum \text{ord}_\tau = k/6$, valid for any modular form of weight $2k$. At weight $2k = 4$ the formula gives $\text{ord}_{\tau_0} = 1$ and there can be no other zero.

²¹The operation S^2 discussed in [6] acts trivially in this circumstance.

of vacua in terms of the extrema of the integrable system, parametrized by the coordinates Z_i (namely, the complexified Wilson lines), at finite coupling τ . That description too can be obtained, but it demands further effort. We can for instance work with the following parameterization of the massless branch

$$\begin{aligned}\mathcal{X}_1 &= \frac{i\sqrt{g_2}}{2\sqrt{3}} \left(\lambda - \frac{1}{\lambda} \right) \\ \mathcal{X}_2 &= \frac{\sqrt{g_2}}{4} \left[\left(\lambda + \frac{1}{\lambda} \right) - \frac{i}{\sqrt{3}} \left(\lambda - \frac{1}{\lambda} \right) \right] \\ \mathcal{X}_3 &= \frac{\sqrt{g_2}}{4} \left[- \left(\lambda + \frac{1}{\lambda} \right) - \frac{i}{\sqrt{3}} \left(\lambda - \frac{1}{\lambda} \right) \right],\end{aligned}\tag{4.17}$$

for $\lambda \in \mathbb{C}^*$. However, we still have to take into account both the fact that we have a double cover (when we solve for \mathcal{Y}_i) as well as the action of the Weyl group to faithfully describe the branch of vacua. The Weyl group has generators that exchange two distinct coordinates, $Z_i \leftrightarrow -Z_j$ (while also changing the sign of the third coordinate). This translates into identifications on our parameter space:

- $Z_2 \leftrightarrow -Z_3$ corresponds to $\lambda \leftrightarrow -\frac{1}{\lambda}$.
- $Z_1 \leftrightarrow -Z_2$ corresponds to $\lambda \leftrightarrow e^{-2\pi i/3}\lambda$.
- $Z_1 \leftrightarrow -Z_3$ corresponds to $\lambda \leftrightarrow e^{2\pi i/3}\lambda$,

and each transformation exchanges the two sheets of the \mathcal{Y} cover. Hence the massless branch is a double cover of the sphere parametrized by λ . We excise the points $\lambda = 0$ as well as the point $\lambda = \infty$, because the superpotential blows up in these points. This indicates the enhancement of gauge symmetry, and the breakdown of the effective superpotential description at these \mathbb{Z}_3 fixed points. A fundamental domain for λ is given by the following region: $|\lambda| \leq 1$ and $\pi/6 \leq \arg \lambda \leq 5\pi/6$ with a \mathbb{Z}_2 identification of the borders of the unit disk as well as of the two rays on the boundary.

We now wish to distinguish between two physically distinct sets of configurations. They are characterized by the way they behave under the charge conjugation symmetry of the gauge theory. Conjugation acts by exchanging $Z_1 \leftrightarrow Z_2$, which is a global symmetry of the gauge theory, inherited by the low-energy effective superpotential. When we have the equality $\mathcal{X}_1 = \mathcal{X}_2$ (or a permutation thereof), we can either have $Z_1 = -Z_2$ or $Z_1 = Z_2$, modulo the periodicity of these variables. The first case corresponds to a fixed point of the local Weyl symmetry group, and it leads to a singular term in the effective superpotential, indicating the enhancement of the gauge group (i.e. the fact that we leave the Coulomb branch). We exclude this singular configuration from our analysis. The second case indicates a fixed point of the charge conjugation symmetry. This occurs when $\lambda^6 = -1$. When there is no equality between any of the variables \mathcal{X}_i , we are at a less symmetric point on the massless branch. These two regimes will lead to a qualitatively different solution for the variables Z_i as we show in detail below.

We would like to solve equation (4.9) for the complexified Wilson lines Z_i . The solution relies on inverting the Weierstrass function. The techniques for performing this inversion

were presented in [39] by Eichler and Zagier in their analysis of the zeros of the Weierstrass \wp function. These authors also study the solutions to the equation $\wp(Z) = \mathcal{X}(\tau)$ where $\mathcal{X}(\tau)$ is a (e.g. meromorphic) modular form of weight 2. Our equation does not fit this mold – the Weierstrass function is equal to the square root of a modular form of weight 4. Still, we can apply the bulk of the Eichler-Zagier methods. The Eichler-Zagier technique for inverting the Weierstrass function consists of two parts. On the one hand, since the argument Z is multi-valued due to the periodicity of the Weierstrass function, it is useful to derive with respect to the modular parameter τ twice, to eliminate this ambiguity. The two integration constants that one subsequently needs can be determined by matching the semi-classical limits. On the other hand, one inverts the equation through integration of the defining equation for the elliptic curve

$$\wp(Z; \tau) = \mathcal{X}(\tau) \quad \Longleftrightarrow \quad Z = \pm \frac{\sqrt{3}}{2\pi} \int_{\frac{3}{\pi^2}\mathcal{X}(\tau)}^{\infty} \frac{dt}{\sqrt{t^3 - 3E_4(\tau)t - 2E_6(\tau)}}. \quad (4.18)$$

From this equation, we determine the second derivative with respect to τ , by multiple application of the Ramanujan identities for the derivatives of the Eisenstein series. The calculation is presented in pedagogical detail in [39] and results in the equality

$$\pm \frac{d^2 Z}{d\tau^2} = \left(4\pi^2(g_3 - 4\mathcal{X}^3 + g_2\mathcal{X})D_3D_6\mathcal{X} + 2\pi^2(12\mathcal{X}^2 - g_2)(D_6\mathcal{X})^2 + (6g_3\mathcal{X} + g_2^2/3)D_6\mathcal{X} \right. \\ \left. + \frac{1}{72\pi^2}(12g_2\mathcal{X}^4 + 3g_2^2\mathcal{X}^2 + 6g_2g_3\mathcal{X} - g_2^3 + 27g_3^2) \right) / (4\mathcal{X}^3 - g_2\mathcal{X} - g_3)^{\frac{3}{2}}, \quad (4.19)$$

where the function \mathcal{X} acts as a seed, and the modular covariant derivative is given by $D_n = q\partial_q - \frac{1}{n}E_2$. The integration constants are fixed by taking the semi-classical limit of the formula (4.18). Here, we will add a point to the analysis in [39], by exhibiting a special case of the limiting formula, which is also physically distinct. We define the variable $\mathcal{X}_{i\infty} = \mathcal{X}(\tau \rightarrow i\infty)$. If $\mathcal{X}_{i\infty} \neq -\frac{\pi^2}{3}$, the semi-classical limit is given by [39]

$$Z(\tau \rightarrow i\infty) = \frac{1}{2} \pm \frac{1}{2\pi i} \log \frac{1 + \sqrt{\frac{2}{3} - \frac{1}{\pi^2}\mathcal{X}_{i\infty}}}{1 - \sqrt{\frac{2}{3} - \frac{1}{\pi^2}\mathcal{X}_{i\infty}}} \quad (4.20)$$

while for the case $\mathcal{X}_{i\infty} = -\frac{\pi^2}{3}$ it is

$$Z(\tau \rightarrow i\infty) = \pm \frac{1}{4} + \frac{\tau}{2}. \quad (4.21)$$

The latter case occurs when the \mathcal{X}_i are at a charge conjugation fixed point, i.e. a fixed point of the global \mathbb{Z}_2 symmetry. Note that the limit formula (3.11) shows that this case is common. Let us nevertheless first concentrate on the case in which all the variables \mathcal{X}_i are different, and construct the solution for the variables Z_i . We then come back to the global \mathbb{Z}_2 fixed point.

The points $\lambda^6 \neq -1$

The point $\lambda = 1$, for instance, is representative for all λ not at a \mathbb{Z}_2 fixed point. In this case, we have

$$\begin{aligned}\mathcal{X}_1 &= 0 \\ \mathcal{X}_2 &= \frac{\sqrt{g_2}}{2} \\ \mathcal{X}_3 &= -\frac{\sqrt{g_2}}{2},\end{aligned}$$

and the formulas (4.20) and (4.19) from [39] apply. We can for instance write the solution as a series expansion q at large imaginary τ

$$\begin{aligned}\pm \pi Z_1 &= \frac{1}{2} (\pi - i \cosh^{-1}(5)) + 36i\sqrt{6}q + 6588i\sqrt{6}q^2 + \dots \\ \pm \pi Z_2 &= \left(\frac{\pi}{2} - i \tanh^{-1} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} \right) \right) + (-36i\sqrt{2} - 18i\sqrt{6})q + (1188i\sqrt{2} - 3294i\sqrt{6})q^2 + \dots \\ \pm \pi Z_3 &= -\pi - \frac{i}{2} \log \left(1 - \frac{6}{3 + \sqrt{6 + 3\sqrt{3}}} \right) + (36i\sqrt{2} - 18i\sqrt{6})q + (-1188i\sqrt{2} - 3294i\sqrt{6})q^2 + \dots\end{aligned}\tag{4.22}$$

The series that we obtain has a finite radius of convergence. The integration formula (4.18) is valid at any modular parameter τ . In this explicit solution (4.22), we can choose a sign for each Z_i , consistently with the constraint $Z_1 + Z_2 + Z_3 \equiv 0$. Thus, we see that we must pick the same sign for all Z_i – there are two solutions. The solutions are invariant under T -duality. This implies that they are also S -invariant, since $1 = (ST)^3 = S^3 = S$. The semi-classical limit of these vacua lies in the class $J_0 = \{1, 2\}$. The semi-classical limit of the massless branch that contains these vacua can be obtained by setting $Y_{1,2} = 0$ and taking the corresponding limit on the equations (4.15) parameterizing the branch.

The \mathbb{Z}_2 symmetric points

We return to the \mathbb{Z}_2 symmetric values of λ which lie at $\lambda^6 = -1$. Let us further concentrate on the case where the equality $\mathcal{X}_2 = \mathcal{X}_3$ holds. Note that the condition we impose is duality invariant. The solutions will therefore transform into each other under the $SL(2, \mathbb{Z})$ action of the duality group. We solve for the coordinates Z_i at these particular points. From equation (4.17) we read that the equality $\mathcal{X}_2 = \mathcal{X}_3$ translates into $\lambda^2 = -1$, which implies that we can focus on the two points $\lambda = \pm i$.

At the value $\lambda = -i$, we have to solve the equations:

$$\mathcal{X}_1 = \sqrt{\frac{g_2}{3}}\tag{4.23}$$

$$\mathcal{X}_2 = -\frac{1}{2}\sqrt{\frac{g_2}{3}}\tag{4.24}$$

$$\mathcal{X}_3 = -\frac{1}{2}\sqrt{\frac{g_2}{3}}.\tag{4.25}$$

We begin with the second equation (4.24), for which the equality (4.21) gives the asymptotic value $Z_2(i\infty) = \pm \frac{1}{4} + \frac{\tau}{2}$. We define

$$Z_2 = \frac{1}{4} + \frac{\tau}{2} - \frac{1}{2}\alpha(q) \quad (4.26)$$

to be the solution of (4.24) such that $\alpha(q)$ has semi-classical behavior

$$\alpha(q) = \frac{8}{\pi}q^{\frac{1}{2}} + O\left(q^{3/2}\right) \quad (4.27)$$

and is analytic along the line $i\mathbb{R}_+^*$. Note that from the equation, we can compute the Fourier expansion to arbitrary order.²² Next, we consider the first equation (4.23). The asymptotic behavior of its solutions $\pm Z_1$ is now given by equation (4.20), and it is $Z_1(i\infty) = \frac{1}{2}$. The exact solution involves the function $\alpha(q)$ just defined, $Z_1 = \frac{1}{2} \pm \alpha(q)$ as a consequence of the doubling formula

$$\wp\left(\frac{1}{2} \pm \alpha\right) = -2\wp\left(\frac{1}{4} + \frac{\tau}{2} - \frac{1}{2}\alpha(q)\right) + \frac{6\wp\left(\frac{1}{4} + \frac{\tau}{2} - \frac{1}{2}\alpha(q)\right)^2 - \frac{1}{2}g_2}{4\wp'\left(\frac{1}{4} + \frac{\tau}{2} - \frac{1}{2}\alpha(q)\right)^2} = \sqrt{\frac{g_2}{3}}. \quad (4.29)$$

The relative sign is determined by the requirement that $Z_3 = -Z_1 - Z_2$ be a solution of (4.25). Therefore we have found two inequivalent vacua at $\lambda = -i$:

$$(Z_1, Z_2) = \left(\frac{1}{2} + \alpha, \frac{1}{4} + \frac{\tau}{2} - \frac{\alpha}{2}\right) \quad (z_1, z_2, z_3) = \left(\frac{3}{4} + \frac{\tau}{2} + \frac{\alpha}{2}, \frac{1}{4} + \frac{\tau}{2} - \frac{\alpha}{2}, 0\right) \quad (4.30)$$

and

$$(Z_1, Z_2) = \left(\frac{1}{2} - \alpha, \frac{1}{4} + \frac{\tau}{2} + \frac{\alpha}{2}\right) \quad (z_1, z_2, z_3) = \left(\frac{3}{4} + \frac{\tau}{2} - \frac{\alpha}{2}, \frac{1}{4} + \frac{\tau}{2} + \frac{\alpha}{2}, 0\right). \quad (4.31)$$

We now turn to the value $\lambda = i$ and proceed similarly. Our task is to solve

$$\mathcal{X}_1 = -\sqrt{\frac{g_2}{3}} \quad (4.32)$$

$$\mathcal{X}_2 = \frac{1}{2}\sqrt{\frac{g_2}{3}} \quad (4.33)$$

$$\mathcal{X}_3 = \frac{1}{2}\sqrt{\frac{g_2}{3}}. \quad (4.34)$$

We define $\frac{1}{2} - \beta(\tau)$ to be the solution of (4.33) with semi-classical behavior

$$\beta(q) = \frac{i}{2\pi} \log\left(2 + \sqrt{3}\right) + O(q) \quad (4.35)$$

²²We have

$$\begin{aligned} \pi\alpha(q) = & 8q^{\frac{1}{2}} - \frac{1088q^{\frac{3}{2}}}{3} + \frac{198288q^{\frac{5}{2}}}{5} - \frac{39006080q^{\frac{7}{2}}}{7} + \frac{7975383560q^{\frac{9}{2}}}{9} \\ & - \frac{1669600216512q^{\frac{11}{2}}}{11} + \frac{355119960987280q^{\frac{13}{2}}}{13} + \dots \end{aligned} \quad (4.28)$$

and demand analyticity on $i\mathbb{R}_+^*$,²³ and again we find that $2\beta(q)$ is a solution of (4.32) using the duplication formula for the Weierstrass function. The signs are determined as previously, and we conclude that a solution is

$$(Z_1, Z_2) = (2\beta, \frac{1}{2} - \beta) \quad (z_1, z_2, z_3) = \left(\frac{1}{2} + \beta, \frac{1}{2} - \beta, 0\right). \quad (4.36)$$

As before, we could flip the sign in front of β in this expression, but this would lead to an equivalent vacuum. We have only one vacuum at $\lambda = i$.

While for generic λ the action of T -duality and as a consequence $SL(2, \mathbb{Z})$ duality on the vacua was trivial, here we see, e.g. from the expansion (4.27), that T -duality exchanges the two vacua (4.30) and (4.31). As a consequence S -duality will act as well. We devote the next paragraph to a detailed study of these dualities.

Dualities at the \mathbb{Z}_2 symmetric points

In the course of our analysis we have found the four solutions of the equation

$$\wp(z)^2 = \frac{g_2}{2} \quad (4.37)$$

that we can gather in a vector

$$V(\tau) = \begin{pmatrix} \frac{1}{2} + \alpha(\tau) \\ \frac{1}{2} - \alpha(\tau) \\ 2\beta(\tau) \\ -2\beta(\tau) \end{pmatrix}, \quad (4.38)$$

which can be interpreted as a vector-valued and multi-valued modular form [39]. The word multi-valued here refers to the fact that these quantities are defined up to periods of the Weierstrass function. This vector transforms under $SL(2, \mathbb{Z})$ according to

$$V(\tau) \xrightarrow{T} V(\tau + 1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V(\tau), \quad (4.39)$$

and

$$V(\tau) \xrightarrow{S} V\left(\frac{-1}{\tau}\right) = \frac{-1}{\tau} \left[\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} V(\tau) - \begin{pmatrix} \tau \\ 0 \\ 0 \\ 1 \end{pmatrix} \right], \quad (4.40)$$

²³We have the further expansion

$$\begin{aligned} i\pi\beta = & -\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) + 12\sqrt{3}(q - 87q^2 + 11080q^3 - 1671095q^4 + \frac{1384694994q^5}{5} \\ & - 48732765432q^6 + \frac{62575601740112q^7}{7} - 1690589139219255q^8 + 327268705474374265q^9 + \dots). \end{aligned}$$

where we have used the analyticity along $i\mathbb{R}_+^*$ to fix the periodic dependence. Thus, we have a weight -1 modular form up to periodicity. Let's call T and S the matrices that appear in these equations and that are associated to the two generators of $SL(2, \mathbb{Z})$. The periodicity is linear in the modular parameter τ , such that again, if we take two derivatives with respect to τ , this ambiguity drops out, and we find a vector valued modular form of weight 3:

$$V''(\tau + 1) = TV''(\tau), \quad V''\left(\frac{-1}{\tau}\right) = (-\tau)^3 SV''(\tau). \quad (4.41)$$

The method of [39] gives the explicit solution

$$\psi(\tau) = V''(\tau) \times \left[\left(\frac{g_2(\tau)}{3} \right)^{\frac{3}{2}} - g_3(\tau) \right]^{\frac{3}{2}} \quad (4.42)$$

and each component of the vector V is given as

$$\pm V'' = \frac{4\pi^2(g_3 - \phi_\pm^3)D_3D_6\phi_\pm + 18\pi^2\phi_\pm^2(D_6\phi_\pm)^2 + 3\phi_\pm(2g_3 + \phi_\pm^3)D_6\phi + \frac{1}{8\pi^2}(4\phi_\pm^6 + 2g_3\phi_\pm^3 + 3g_3^2)}{(\phi_\pm^3 - g_3)^{\frac{3}{2}}}$$

where the seed ϕ_\pm is a branch of the square root of the Eisenstein series $\phi_\pm = \pm\sqrt{\frac{g_2}{3}}$, and the 4 components of V'' correspond to the 4 possible choices of signs (on the left, and on the right hand side independently). We give the first few terms (the first line is obtained from ϕ_+ and the second line from ϕ_-):

$$V''(\tau) = \pm 8\pi \left(\frac{q^{\frac{1}{2}} - 408q^{\frac{3}{2}} + 123930q^{5/2} - 34130320q^{7/2}}{12i\sqrt{3}(q - 348q^2 + 99720q^3 - 26737520q^4 + \dots)} \right) = \pm \begin{pmatrix} \alpha'' \\ 2\beta'' \end{pmatrix} \quad (4.43)$$

and

$$\psi(\tau) \propto -q^2 + 336q^3 - 94824q^4 + 25238080q^5 - 6506938620q^6 + \dots \quad (4.44)$$

After double integration, this characterizes the q expansion of α, β , and therefore analytically completes the series we obtained previously. We further analytically continue the functions α and β in the double cover of the upper half plane. The triplet of solutions to the equation becomes degenerate at the zeros of E_4 . Note that we can switch branch for the seed by rotating around the zero $\tau_0 = e^{2\pi i/3}$ of the weight 4 Eisenstein series. As a consequence, this operation flips α and β , and this introduces a monodromy amongst the sheets of massless vacua in the elliptic integrable system parameterization.

Summary Remarks

We recapitulate the duality diagram for both the massive and massless extrema of the $su(3)$ integrable system.²⁴ We have four massive vacua, of which two are self- S -dual, and two are mutually S -dual. They form a singlet and a triplet under T -duality. We have one

²⁴ We repeat that the global aspects of the gauge group can be taken into account by carefully treating the subgroup of \mathbb{Z}_3 which one chooses as center, and the possible electro-magnetic line operators in the theory, which have consequences on the periodic identifications of variables.

massless branch which is duality invariant in the elliptic curve variables.²⁵ We note that the semi-classical limit that allows for the Higgs vacuum, also sees the massless branch.

Partition	J_0	Unbroken	Vacua
$1 + 1 + 1$	\emptyset	$su(3)$	3 confining vacua of pure $\mathcal{N} = 1$
$2 + 1$	$\{1\}$ or $\{2\}$	$u(1)$	the massless branch
3	$\{1, 2\}$	1	1 Higgs vacuum + the massless branch

Table 4: Summary of vacua for $su(3)$

There is a more intricate description of the massless branch in terms of the elliptic integrable system variables, which allows to follow the duality map on the massless vacua point by point. For the extremal positions of the massless vacua in terms of the complexified Wilson lines, we have exhibited a point of monodromy on the boundary of the fundamental domain, and in particular, the elliptic point of order 3 of the $SL(2, \mathbb{Z})$ action on the upper half plane. This point is a singular point for the manifold of massless vacua. It is reminiscent of the point of monodromy in the interior of the fundamental domain for two massive vacua of the $so(8)$ theory [17].

4.3 The Gauge Algebra $su(4)$

We have obtained a complete picture of the massive and massless vacua of the $su(3)$ theory. In this subsection, dedicated to the gauge algebra $su(4)$, we will only perform a partial analysis. Recall that for $su(4)$, the partition $1 + 1 + 1 + 1$ gives rise to an affine Toda limit with four solutions, which correspond to the four confining vacua of pure $\mathcal{N} = 1$. The partition $2 + 2$ corresponds to the choice $J_0 = \{\alpha_1, \alpha_3\}$ which gives rise to two trigonometric A_1 systems with one solution, and the two remaining variables then form an A_2 affine Toda system which has two solutions, corresponding to the two confining vacua of the unbroken $su(2)$ gauge algebra. Finally, we have the partition 4 which corresponds to the trigonometric A_3 system. This gives rise to a real extremum which represents the fully Higgsed vacuum. We have a total of seven massive vacua.²⁶ Our focus in the following are massless vacua. A natural way to generate massless vacua is by exploring the partitions $2 + 1 + 1$ and $3 + 1$ which leave unbroken abelian gauge group factors. We will consider them in turn. Let us first remind the reader that the superpotential for the $su(4)$ gauge algebra is

$$\mathcal{W}_{A_3}(Z_1, Z_2, Z_3) = \wp(Z_1) + \wp(Z_2) + \wp(Z_3) + \wp(Z_1 + Z_2) + \wp(Z_2 + Z_3) + \wp(Z_1 + Z_2 + Z_3), \quad (4.45)$$

in variables Z_i which are coefficients of fundamental weights.

The Partition $2 + 1 + 1$

The partition $2 + 1 + 1$ corresponds to a choice of simple root system $J_0 = \{1\}$. The centralizer algebra is $su(2) \oplus u(1)$ in this case. We may intuit the existence of two massless

²⁵There is a point this branch which is S-duality and T-duality invariant. It is given by $(z_1, z_2, z_3) = (1/2, \tau/2, (1 + \tau)/2)$, and is mentioned in [31].

²⁶There are other complex extrema of the trigonometric integrable systems.

branches on the basis of this centralizer algebra. We will approach them through the semi-classical limit.

In this limit, we have one leading trigonometric root that sets $Y_1 = 0$. To find the other shifts, we use a heuristic argument based on cancellations that happen in the superpotential at first order in perturbation theory, where $X_1 = \frac{1}{2}$. Such cancellations occur at level $n = 1$ in (3.28) in the sum over the roots that have a non-vanishing scalar product with \hat{Y} . As illustrated on the affine Dynkin diagram on figure 5, the contributions of α_0 and $\alpha_0 + \alpha_1$ will cancel each other in (3.30), as well as α_2 and $\alpha_1 + \alpha_2$, and all other roots involving α_0 and α_2 are suppressed in the semi-classical limit. Therefore in order to stabilize the system we use the next level $n = 2$ for these roots, which then contribute with factors of q^{2Y_0} and q^{2Y_2} . On the other hand α_3 contributes with a factor q^{Y_3} . Stabilization at leading order requires that these powers of q be equal, and we therefore propose the following ansatz:

$$\begin{cases} Y_1 = 0 \\ 2Y_0 = 2Y_2 = Y_3 \\ Y_0 + Y_1 + Y_2 + Y_3 = 1 \end{cases} \implies \begin{cases} Y_0 = \frac{1}{4} \\ Y_2 = \frac{1}{4} \\ Y_3 = \frac{1}{2} \end{cases} . \quad (4.46)$$

To obtain the subleading Toda potential, we need to take into account the non-perturbative corrections to the value $X_1 = \frac{1}{2}$. Firstly, we expand the superpotential (4.45) around the leading order values (4.46), assuming that the variation δX_1 of X_1 behaves as a power of q . The dominant terms are

$$\frac{1}{\pi^2} \mathcal{W} \left(\frac{1}{2} + \delta X_1, \frac{\tau}{4} + X_2, \frac{\tau}{2} + X_3 \right) = -1 + \pi^2 \delta X_1^2 + 8i\pi \delta X_1 q^{\frac{1}{4}} (e^{2i\pi X_2} - e^{-2i\pi X_2 - 2i\pi X_3}) . \quad (4.47)$$

There is a linear term in the non-perturbative correction δX_1 which determines its value at order $q^{\frac{1}{4}}$:

$$(\delta X_1)_{\frac{1}{4}} = \frac{4i}{\pi} q^{\frac{1}{4}} (e^{-2i\pi(X_2+X_3)} - e^{2i\pi X_2}) . \quad (4.48)$$

This confirms that the value $X_1 = 1/2$ has to be corrected, and that the superpotential should be expanded around the point shifted by $(\delta X_1)_{\frac{1}{4}}$:

$$\frac{1}{\pi^2} \mathcal{W} \left(\frac{1}{2} + (\delta X_1)_{\frac{1}{4}} + \delta X_1, \frac{\tau}{4} + X_2, \frac{\tau}{2} + X_3 \right) = -1 - 4q^{\frac{1}{2}} (9e^{-2\pi i X_3} + e^{2\pi i X_3}) . \quad (4.49)$$

We conclude that X_3 can be determined at this step, and we find

$$X_3 = -\frac{i \log 3}{2\pi} + \frac{1}{2} \mathbb{Z} . \quad (4.50)$$

A longer calculation at higher order shows that X_3 in turn receives non-perturbative corrections, starting at order $q^{\frac{1}{2}}$. Taking into account this second step in our non-perturbative staircase, we find that the superpotential becomes independent of X_2 , and equal to $-1 \mp 24\sqrt{q} - 24q + \dots = \pi^2 E_{2,2}(\pm q^{\frac{1}{2}})$ where the upper sign is for the choice of an integer in equation (4.50) and the lower sign for a strictly half-integer choice. Thus, we have found semi-classical evidence for two one-dimensional complex manifolds of massless vacua characterized by these superpotentials. Again, numerical and analytical evidence can be amassed to argue that the superpotentials are exact.²⁷

²⁷One extra technique compared to those presented elsewhere is to find a special point on the branch,

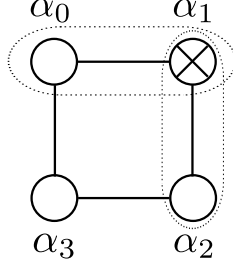


Figure 5: Affine Dynkin diagram for algebra A_3 and partition $2 + 1 + 1$. The crossed simple root corresponds to the set $J_0 = \{1\}$, and the dotted lines encircle roots that cancel the contribution of the simple roots α_0 and α_2 .

The Partition $3 + 1$

The partition $3 + 1$ corresponds to the choice of simple roots $J_0 = \{1, 2\}$. The unbroken gauge algebra is $u(1)$, and we expect one massless branch. Our ansatz for the linear behavior in τ is dictated by the choice of the partition which gives $Y_1 = Y_2 = 0$ and by the symmetry of the affine Dynkin diagram which leads to $Y_3 = Y_0$. Together with the normalization (3.22) we obtain $Y_3 = \frac{1}{2}$. The trigonometric A_2 system arises, and we consider the standard Higgs solution of this system. We thus have $X_1 = 1/3$ and $X_2 = 1/3$ to leading order. We obtain a series of non-perturbative corrections to both leading coordinates and find that when we take those into account, the third variable X_3 parametrizes a massless branch. We calculate the superpotential to order q^2 and it is consistent with the exact value we propose, namely $\mathcal{W} = -2\pi^2 E_{2,2}(q)$.

The Duality Diagram

We have gathered semi-classical and exact data on the $su(4)$ $\mathcal{N} = 1^*$ theory. The duality diagram for the massive states is essentially known, with or without the refinement due to the global choice of gauge group and line operator spectrum. The massless branches fit into the following scheme: we have two massless branches that arise from the partition $2 + 1 + 1$ and they are T-dual. This is consistent with the confining dynamics of the summand $su(2)$ in the unbroken gauge group. The branch that we found for partition $3 + 1$ is self-T-dual. Moreover, the branch with superpotential $\pi^2 E_{2,2}(q^{\frac{1}{2}}) = -\pi^2 (\theta_2(q)^4 + \theta_3(q)^4)$ is S-dual to the manifold with superpotential $\pi^2 (\theta_4(q)^4 + \theta_3(q)^4) = -2\pi^2 E_{2,2}(q)$. Similarly, the branch with superpotential $\pi^2 E_{2,2}(-q^{\frac{1}{2}})$ is self-S-dual. This is a familiar three-node permutation

and then prove that at that point the superpotential takes the claimed value. For the case at hand, for instance, we can concentrate on the point

$$(Z_1, Z_2, Z_3) = \left(\frac{1}{2}, \frac{\tau}{4} + \frac{\gamma}{2}, \frac{\tau}{2} - \gamma \right). \quad (4.51)$$

One then shows that these positions are indeed extremal provided the function $\gamma(\tau)$ satisfies the equation

$$\wp'(2A) + \wp'_2(A) = 0 \quad \text{with} \quad A = \frac{\tau}{4} + \frac{\gamma}{2}. \quad (4.52)$$

One can then also analytically prove that this vacuum is massless and has the claimed superpotential. The result is then valid along the whole branch.

representation of the $SL(2, \mathbb{Z})$ duality group. The table below gives a summary of some of the data we laid bare.

Partition	J_0	Unbroken	Vacua
$1 + 1 + 1 + 1$	\emptyset	$su(4)$	4 confining vacua of pure $\mathcal{N} = 1$
$2 + 1 + 1$	$\{1\}, \{2\}$ or $\{3\}$	$su(2) \oplus u(1)$	branches $\pi^2 E_{2,2}(\pm q^{\frac{1}{2}})$
$3 + 1$	$\{1, 2\}$ or $\{2, 3\}$	$u(1)$	branch $-2\pi^2 E_{2,2}(q)$
$2 + 2$	$\{1, 3\}$	$su(2)$	2 massive vacua
4	$\{1, 2, 3\}$	1	1 Higgs massive vacuum

Table 5: Summary of vacua for $su(4)$

Summary Remarks

We again found new features in the $su(4)$ analysis. These included a staircase structure for determining the positions, with each step corresponding to non-perturbative corrections of a given order. We also discovered an example in which massless manifolds fit into a non-trivial duality diagram. These features are expected to be generic. We moreover are bound to find higher dimensional vacuum manifolds when higher dimensional abelian centralizers are present.

It would be interesting to fully complete the analysis of the vacua of the $su(4)$ theory, in the spirit of the analysis we performed for $su(3)$. In particular, one can exploit the algebraic approach, and parametrize the extrema in terms of algebraic equations. This will allow to determine for instance potential singularities, and possible intersections of manifolds of massless vacua for given values of the complexified coupling τ .

4.4 A Word on the $su(N)$ Theory

It should be clear that in the $su(N)$ case there will be many extra solutions compared to the known sublattices of order N of the torus that represent the massive vacua. Below, we offer only one rudimentary observation on the massless vacua.

The number of massless directions at $\tau \rightarrow i\infty$ in the integrable system equals the number of $U(1)$ factors for the semi-classical vacuum in the Coulomb phase. Indeed, all directions that are stabilized by terms with leading behavior a power of q will become untethered when we take the semi-classical limit. In our problem, these are all directions associated to the (affine) Toda system(s). Thus, in the semi-classical limit, we obtain $|\bar{J}_0| - 1$ flat directions. We can check that this matches the dimension of the semi-classically unbroken abelian factors, when we go to the Coulomb branch.

Recall that a partition (d_i) (which satisfied $\sum d_i = N$) corresponds to strands $(d_i - 1)$ in the set of simple roots of A_{N-1} . The number of coordinates we fix at leading order, using the trigonometric integrable system potential, is equal to $\sum_i (d_i - 1)$. The number of coordinates that is unfixed then, in the semi-classical limit (taken on the low-energy effective action) is equal to $N - 1 - \sum_i (d_i - 1) = |i| - 1$ where $|i|$ counts the number of (non-zero) terms in the partition.

The number of abelian factors in the Coulomb phase is given by the rank of the centralizer of the nilpotent orbit. The rank is equal to $\sum_j (r_j - 1) + k$ where the r_j are defined as the number of times the summand j appears in the partition and $k = |\{i | r_i > 0\}| - 1$ as in the discussion around (4.3). If we compute this sum, using $\sum_j r_j = |i|$ and $\sum_j 1 = k + 1$, we find $|i| - 1$, which matches the number of massless modes in the semi-classical limit.

5 Discrete Gauge Groups, Wilson Lines and the G_2 Theory

In section 4 we described features of semi-classical limits and massless vacua for the $\mathcal{N} = 1^*$ theory with $su(N)$ gauge algebra. We now wish to examine more closely another feature that we encountered in the example of $so(5)$ put forward in section 2. We study the appearance of extra massive vacua that occur on $\mathbb{R}^{2,1} \times S^1$, coded in advanced nilpotent orbit theory. We classified classical vacua of $\mathcal{N} = 1^*$ theory on \mathbb{R}^4 using nilpotent orbit theory in [9]. In this subsection, we wish to stress an important extra feature that comes into play after compactifying the theory on a circle, namely the multiplication of massive vacua through the existence of discrete gauge groups and Wilson lines.

5.1 Discrete Gauge Groups and Wilson Lines

We have a space-time equal to $\mathbb{R}^{2,1} \times S^1$, and parametrize the circle by the coordinate θ with period 2π , and we denote by R the radius of the circle. Suppose we fix constant vacuum expectation values $X^j(0)$ for the three $\mathcal{N} = 1$ adjoint chiral multiplets ($j = 1, 2, 3$). Moreover, we have them satisfy $su(2)$ commutation relations, as required for constant scalar field configurations to obey the F-term equations of motion. Let us further suppose that the chosen $su(2)$ algebra has a discrete centralizer (equal by assumption to the component group of the centralizer). We therefore have a discrete unbroken gauge group.

It should be clear that a discrete component group permits discrete Wilson lines upon compactification. Suppose that a discrete centralizer of the $sl(2)$ triple contains a non-trivial element $e^{2\pi i a}$ with a an element in the Lie algebra \mathfrak{g} of the gauge group. Then we can propose semi-classical configurations that are new to the theory compactified on the circle, compared to the theory on \mathbb{R}^4 . These configurations are:²⁸

$$X^j(\theta) = \exp(ia\theta)X^j(0)\exp(-ia\theta). \quad (5.1)$$

The gauge field component along the circle is fixed to be the constant $A_\theta = \frac{1}{R}a$. These configurations are covariantly constant, since the gauge covariant derivative is given by $D_\theta X^j = \partial_\theta X^j - i[a, X^j]$, and they are periodic by the fact that the group element $e^{2\pi i a}$ belongs to the centralizer of $X^j(0)$:

$$\begin{aligned} D_\theta X^j(\theta) &= 0 \\ X^j(\theta + 2\pi) &= X^j(\theta). \end{aligned} \quad (5.2)$$

²⁸Note that we satisfy ordinary boundary conditions. Interesting boundary conditions twisted by outer automorphisms can be imposed for gauge algebras of type A , D and E_6 . See e.g. [36, 37].

By construction these configurations satisfy the F-term equations of motion. Thus, on $\mathbb{R}^{2,1} \times S^1$, the unbroken discrete gauge group gives rise to a larger set of semi-classical configurations. A formal, non-periodic gauge transformation transforms the solutions $X^j(\theta)$ with non-zero Wilson line into the configurations $X^j(0)$ with zero Wilson line. Needless to say, these configurations remain physically distinct on S^1 . Furthermore, true gauge transformations with constant parameter transform the constant gauge field a within a given conjugacy class. Thus, for each non-trivial conjugacy class in the discrete gauge group, we find a new semi-classical configuration on the circle.

For a purely discrete centralizer, the above discussion is complete. When there are both continuous identity components and a discrete component group, the analysis requires more care. Note for instance that the role of the component group can also be to exchange continuous factors in the centralizer, as discussed in detail in [9], or to break an abelian factor in the centralizer as illustrated in [9] and section 2.

5.2 The Semi-Classical Vacua for G_2

We will discuss in greater detail an example theory that illustrates the above configurations neatly, namely $\mathcal{N} = 1^*$ theory with gauge algebra G_2 . We start out with a description of our semi-classical expectations. We will see that G_2 is a good testing ground for the above general discussion. We perform semi-classical limits on the low-energy effective potential, and compare the results to our semi-classical expectations for the gauge theory. The extra configurations described above will indeed appear as solutions. We conclude with a duality diagram for the vacua, a point of monodromy, and other findings on the gauge theory and integrable system that are of interest.

Firstly, let's recall the classification of semi-classical configurations for $\mathcal{N} = 1^*$ theory with gauge group G_2 on \mathbb{R}^4 . The group G_2 is both connected and simply-connected. For the $\mathcal{N} = 1^*$ theory on \mathbb{R}^4 , we classify semi-classical configurations by enumerating embeddings $X^j : \mathfrak{sl}_2 \rightarrow G_2$, which are in one-to-one correspondence with nilpotent orbits of the Lie algebra G_2 . Again, we apply the classification theory of Bala-Carter [18, 19] and Sommers [20]. We pause for a while to explain how this classification is obtained.

Bala-Carter Theory for Nilpotent Orbits

Suppose we want to find the nilpotent orbits of a Lie algebra \mathfrak{g} . The Bala-Carter theorem states that this is equivalent to finding the pairs $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ where \mathfrak{l} is a Levi subalgebra of \mathfrak{g} and $\mathfrak{p}_{\mathfrak{l}}$ is a distinguished parabolic subalgebra of $[\mathfrak{l}, \mathfrak{l}]$. In order to fully understand this statement, we recall three useful definitions and properties:

- A parabolic subalgebra of \mathfrak{g} is a subalgebra which is conjugate to a subalgebra of the form \mathfrak{p}_{Θ} where Θ is a set of simple roots, and where \mathfrak{p}_{Θ} is generated by
 - (a) The Cartan subalgebra ;
 - (b) The root spaces corresponding to the root system $\langle \Theta \rangle$ created by Θ ;
 - (c) The root spaces corresponding to all other positive roots.

We have that \mathfrak{p}_{Θ} and $\mathfrak{p}_{\Theta'}$ are conjugate if and only if $\Theta = \Theta'$.

- We can decompose a parabolic subalgebra $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta$, where the part generated by points (a) and (b) above is the Levi subalgebra \mathfrak{l}_Θ , and the part generated by point (c) is the nilradical \mathfrak{n}_Θ . The algebras \mathfrak{l}_Θ and $\mathfrak{l}_{\Theta'}$ are conjugate if and only if $\langle \Theta \rangle$ and $\langle \Theta' \rangle$ are Weyl-conjugate.
- A parabolic subalgebra $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta$ is *distinguished* if and only if $\dim \mathfrak{l}_\Theta = \dim \mathfrak{n}_\Theta / [\mathfrak{n}_\Theta, \mathfrak{n}_\Theta]$.

Let's apply this to G_2 . In table 6 we compute, for the 4 conjugacy classes of parabolic subalgebras, the dimensions of the corresponding Levi subalgebra and of the nilradical. This gives the list of Levi subalgebras, which is the first step of the classification. The second step is to find, for each Levi subalgebra, the distinguished parabolic subalgebras of $[\mathfrak{l}_\Theta, \mathfrak{l}_\Theta]$. This is trivial for 0, \tilde{A}_1 and A_1 , in which one can check that there is exactly one distinguished parabolic subalgebra, while for G_2 we use again table 6 in which we read that there are two distinguished parabolic subalgebras.

Θ	$\dim \mathfrak{l}_\Theta$	$\dim \mathfrak{n}_\Theta$	$\dim \mathfrak{n}_\Theta / [\mathfrak{n}_\Theta, \mathfrak{n}_\Theta]$	$[\mathfrak{l}_\Theta, \mathfrak{l}_\Theta]$
\emptyset	2	6	2	0
$\{\alpha_1\}$	4	5	4	\tilde{A}_1
$\{\alpha_2\}$	4	5	2	A_1
$\{\alpha_1, \alpha_2\}$	14	0	0	G_2

Table 6: The 4 (conjugacy classes of) parabolic subalgebras of G_2 , which are in one-to-one correspondence with subsets of the set of simple roots. We read that a given parabolic subalgebra is distinguished if and only if the numbers in the second and fourth columns are equal.

We conclude that there are 5 nilpotent orbits in the G_2 Lie algebra, which are summarized in table 7.

Θ	Name and Number
\emptyset	0 \rightarrow 1 orbit
$\{\alpha_1\}$	$\tilde{A}_1 \rightarrow$ 1 orbit
$\{\alpha_2\}$	$A_1 \rightarrow$ 1 orbit
$\{\alpha_1, \alpha_2\}$	$G_2 \rightarrow$ 2 orbits called G_2 and $G_2(a_1)$

Table 7: The 4 (conjugacy classes of) parabolic subalgebras of G_2 , which are in one-to-one correspondence with subsets of the set of simple roots.

Sommers generalizes this classification by allowing Θ to be a proper subset of the set of affine simple root, and calling *pseudo-Levi* subalgebras the additional generated algebras [20]. The main theorem states that there is a bijection between

- (i) Conjugacy classes of pairs (X, C) , where X is a nilpotent element and C is a conjugacy class in the component group of the centralizer and

- (ii) Conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ where \mathfrak{l} is a pseudo-Levi subalgebra and $\mathfrak{p}_{\mathfrak{l}}$ is a distinguished parabolic subalgebra of $[\mathfrak{l}, \mathfrak{l}]$.

As a result, we find that for G_2 there are 7 such conjugacy classes, captured in table 8. The centralizer of orbit $G_2(a_1)$ (which is a discrete group) has 3 conjugacy classes, and is in fact S_3 , while the other orbits have trivial component group.

W-classes of Θ	$[\mathfrak{l}_{\Theta}, \mathfrak{l}_{\Theta}]$	Distinguished	Orbit	C. C.	Comm.
\emptyset	0	1	1	1	G_2
$\{\alpha_0\}, \{\alpha_2\}$	A_1	Principal A_1	A_1	1	\tilde{A}_1
$\{\alpha_1\}$	\tilde{A}_1	Principal \tilde{A}_1	\tilde{A}_1	1	A_1
$\{\alpha_0, \alpha_1\}$	$A_1 + \tilde{A}_1$	Principal $A_1 + \tilde{A}_1$	$G_2(a_1)$	(12)	1
$\{\alpha_0, \alpha_2\}$	A_2	Principal A_2	$G_2(a_1)$	(123)	1
$\{\alpha_1, \alpha_2\}$	G_2	G_2	G_2	1	1
		$G_2(a_1)$	$G_2(a_1)$	1	1

Table 8: The 7 classes of pairs (X, C) where O is a nilpotent orbit and C is a conjugacy class in the component group of the centralizer of O . We tabulate the (derived algebra of the) pseudo-Levi subalgebra, its distinguished orbits, their name, the conjugacy class and the reductive part of the Lie algebra commutant. The discrete centralizer for the orbit $G_2(a_1)$ is the group S_3 .

Finally, we can apply our reasoning on the multiplication of semi-classical vacua when we compactify the $\mathcal{N} = 1^*$ theory on $\mathbb{R}^{2,1} \times S^1$ with gauge group G_2 . For each nilpotent orbit for which we have a single conjugacy class in the component group, we apply the same reasoning as on \mathbb{R}^4 based on the idea that we obtain pure $\mathcal{N} = 1$ super Yang-Mills theories with a number of massive vacua equal to the dual Coxeter number of the gauge group. We find $4 + 2 + 2 + 1 = 9$ massive vacua in this manner. Moreover, for the nilpotent orbit $G_2(a_1)$, we have three conjugacy classes in the discrete centralizer S_3 , and we therefore expect 3 vacua. We therefore find a total of

$$4 + 2 + 2 + 1 + 3 = 12 \tag{5.3}$$

massive vacua for G_2 .

Similarly, for the semi-classical limits, we have the following expectations. The trigonometric system will be determined by the gauge group breaking pattern, and more specifically by the root system associated to the breaking. We therefore may expect a trigonometric system of type $G_2, A_2, A_1 + \tilde{A}_1, \tilde{A}_1, A_1$ and none at all in the above cases (read from bottom to top). The A_1 and \tilde{A}_1 cases are cases of oblique confinement, and the case of trivial orbit, where the full G_2 gauge group remains unbroken, corresponds to confining vacua. The first three cases are Higgsed vacua, possibly with non-trivial Wilson lines corresponding to the non-trivial conjugacy classes in the component group. We distinguish three different cases, namely zero Wilson line, a Wilson line 2-cycle and a Wilson line 3-cycle since these are the conjugacy classes of the S_3 component group. Note that the two

Higgs vacua associated to the G_2 orbit and $G_2(a_1)$ orbit with trivial conjugacy class share the same symmetry breaking pattern. Although the vacuum expectation values X^j are different, the integrable system may not distinguish them. Taking this last subtlety into account, we can predict that the integrable system has 11 extrema, which are recovered from the semi-classical limit in section 5.3 and found numerically in subsection 5.4.

5.3 The Elliptic Integrable System and the Semi-classical Limits

In this subsection, we explicitly calculate the semi-classical limits on the low-energy effective superpotential and compare the results to our expectations. The effective superpotential that we work with is

$$\mathcal{W}_{G_2,tw}(Z) = \sum_{\alpha \in \Delta_{\text{long}}^+} \wp(\alpha \cdot Z) + \frac{1}{3} \sum_{\alpha \in \Delta_{\text{short}}^+} \wp_3(\alpha \cdot Z). \quad (5.4)$$

We refer to appendix A for our conventions for the root system, weight system, and other Lie algebra data. We can parametrize $Z = 3(z_1\epsilon_1 + z_2\epsilon_2)$ such that

$$\begin{aligned} \mathcal{W}_{G_2,tw}(z_1, z_2) = & \wp(3z_1 - 3z_2) + \wp(3z_1) + \wp(3z_2) \\ & + \frac{1}{3} [\wp_3(z_1 + z_2) + \wp_3(2z_1 - z_2) + \wp_3(-z_1 + 2z_2)], \end{aligned} \quad (5.5)$$

or use the alternative parameterization $Z = Z_1\pi_1 + Z_2\pi_2 = (Z_1 + 2Z_2)\epsilon_1 + Z_2\epsilon_2$. The link is

$$\begin{cases} z_1 = \frac{1}{3}(Z_1 + 2Z_2) \\ z_2 = \frac{1}{3}Z_2 \end{cases} \quad \begin{cases} Z_1 = 3z_1 - 6z_2 \\ Z_2 = 3z_2, \end{cases} \quad (5.6)$$

and the explicit form of the superpotential is then

$$\begin{aligned} \mathcal{W}_{g_2,tw}(Z_1, Z_2) = & \wp(Z_2) + \wp(Z_1 + Z_2) + \wp(Z_1 + 2Z_2) \\ & + \frac{1}{3} [\wp_3(Z_1/3) + \wp_3(Z_1/3 + Z_2) + \wp_3(2Z_1/3 + Z_2)], \end{aligned} \quad (5.7)$$

with $Z_i = X_i + \tau Y_i$. We still have to specify the periodicities and identifications. In the ω_1 direction, we identify by shifts by the weight lattice, and in the ω_2 direction by the co-weight lattice. This implies :

$$(z_1, z_2) \sim \left(z_1 + \frac{2}{3}\omega_1, z_2\right) \sim \left(z_1, z_2 + \frac{2}{3}\omega_1\right) \quad (5.8)$$

and

$$(z_1, z_2) \sim (z_1 + 2\omega_2, z_2) \sim (z_1, z_2 + 2\omega_2) \sim \left(z_1 - \frac{2}{3}\omega_2, z_2 + \frac{2}{3}\omega_2\right). \quad (5.9)$$

The Weyl group action yields the further equivalences

$$(z_1, z_2) \sim (z_2, z_1) \sim (-z_1, -z_2) \sim (z_1, z_1 - z_2). \quad (5.10)$$

We note that the group G_2 has trivial center.

Below, we distinguish between the trigonometric or Higgs limits, in which the leading trigonometric system is of rank two, the oblique limits, in which it is of rank one, and the affine Toda, or confining limit.

5.3.1 The Higgsed Limits

Firstly, we describe the limit for the Higgs vacuum, the 2-cycle vacuum and the 3-cycle vacuum. The superpotential becomes the trigonometric system corresponding to the pseudo-Levi subalgebra.

- **The trigonometric G_2 limit**

In the first $\tau \rightarrow i\infty$ limit, where we take the Wilson line to be $a = (0, 0)$ and consequently $Y_1 = 0 = Y_2$, we find the trigonometric G_2 model for the choice of simple roots $J_0 = \{\alpha_1, \alpha_2\}$:

$$\mathcal{W}_{G_2, tw}(X) \rightarrow \mathcal{W}_{\text{trig}, G_2}. \quad (5.11)$$

We find a real extremum. It can be described through zeroes of an orthogonal polynomial [27, 28].

- **The trigonometric A_2 limit**

In the second limit, corresponding to the 3-cycle conjugacy class and Wilson line $a = (1/3, 0)$ we find the trigonometric A_2 system corresponding to the choice of simple root system $J_0 = \{\alpha_0, \alpha_2\}$. The co-marks give the constraint $1 = Y_0 + 2Y_2 + Y_1$. If we impose $Y_0 = 0 = Y_2$, we find $Y_1 = 1$. We are left with a trigonometric A_2 system corresponding to all the long roots. The extremal positions are therefore given by the equilibria of the trigonometric A_2 integrable system. There is a massive extremum at $(z_1, z_2) = (1/3 + \tau/3, 2/9)$.²⁹

- **Trigonometric $A_1 + \tilde{A}_1$**

The third Higgs vacuum is associated to the Wilson line $a = (1/2, 0)$, with the choice $J_0 = \{\alpha_0, \alpha_1\}$, and gives rise to the trigonometric $A_1 + \tilde{A}_1$ system (with differing coupling constants). One finds a unique extremum up to equivalences, namely $(z_1, z_2) = (\frac{1}{6} + \frac{\tau}{2}, \frac{1}{6})$.

Remark

We remark that it is the centralizer of the Wilson line group element that determines the non-perturbative contributions to the superpotential in the semi-classical limit. Namely, the allowed monopole charges corresponds precisely to positive roots of the pseudo-Levi subalgebra. We recall that we have two configurations in which the full gauge group is broken, namely the orbit labelled G_2 and the orbit labelled $G_2(a_1)$ with zero Wilson line. In the elliptic integrable system, we only identified one real extremum. The two orbits are distinguished through their scalar adjoint vacuum expectation values.

²⁹The A_2 trigonometric model also allows for massless complexified extrema at zeroth order. However, these extrema do not survive the order q perturbation.

5.3.2 The Confining Limit

If we pick zero simple roots, we obtain the affine Toda potential for the algebra $D_4^{(3)} = (G_2^{(1)})^\vee$

$$\mathcal{W}_{G_2,tw}(x_1 + \frac{\tau}{4}, x_2 + \frac{\tau}{12}) = q^{\frac{1}{4}} \left(e^{-6\pi i x_1} + e^{6\pi i x_2} + 3e^{6\pi i(x_1 - 2x_2)} \right) + \dots \quad (5.12)$$

The associated simple roots are α_0 , α_2 and $3\alpha_1$. The extrema of the affine Toda potential can be obtained analytically (see e.g. [38]).

5.3.3 The Oblique Limits

Let's turn to the limits with partial breaking of the gauge algebra through adjoint vacuum expectation values.

- **The Oblique Limit $J_0 = \{1\}$**

The limit $J_0 = \{1\}$ corresponds to the orbit \tilde{A}_1 with unbroken gauge group A_1 . We first determine the non-perturbative corrections to the leading coordinate Z_1 , and find

$$(Z_1, Z_2) = \left(\frac{1}{2} - \frac{4i}{3\pi} e^{2i\pi\delta_2} q^{\frac{1}{4}} + \delta_1, \frac{\tau}{4} + \delta_2 \right), \quad (5.13)$$

and a final stabilized potential (at $\delta_1 = 0$)

$$\mathcal{W}_{G_2,tw}(\delta_2) = \pi^2 \left(1 + 4e^{-4i\pi\delta_2} q^{\frac{1}{2}} - \frac{20}{3} e^{4i\pi\delta_2} q^{\frac{1}{2}} + O(q) \right). \quad (5.14)$$

We can solve for the fluctuation δ_2 using this superpotential, and then find the superpotential at equilibrium to order $q^{\frac{1}{2}}$

$$\mathcal{W}_{G_2,tw} = \pi^2 \left(1 - 8i \sqrt{\frac{5}{3}} \sqrt{q} + \dots \right). \quad (5.15)$$

- **The Oblique Limit $J_0 = \{2\}$**

If we put $Y_1 = \frac{3}{4}, Y_2 = 0$, we get stabilization at level $q^{\frac{3}{2}}$. The first orders in the expansion of the coordinate Z_2 are given by

$$Z_2 = \frac{1}{2} - \frac{4ie^{-2i\pi\delta X_1} \sqrt[4]{q}}{\pi} + \frac{48ie^{-4i\pi\delta X_1} q^{\frac{1}{2}}}{\pi} + \dots, \quad (5.16)$$

to finally find stabilization for the coordinate Z_1 at

$$Z_1 = \frac{3\tau}{4} - \frac{i \log 3}{4\pi} \quad \text{or} \quad \frac{3\tau}{4} - \frac{1}{4} - \frac{i \log 3}{4\pi}. \quad (5.17)$$

The resulting superpotentials in the two inequivalent vacua are

$$\mathcal{W}_{G_2,tw} = \pi^2 (-1 + 312q \pm 5832q^{\frac{3}{2}} + \dots). \quad (5.18)$$

The stabilizing potential for Z_1 arises at sixth order in the non-perturbative expansion parameter $q^{\frac{1}{4}}$.

Remark

One can ask about the oblique limit $J_0 = \{0\}$. We have found no choice of Y_i consistent with the condition $Y_0 = 0$ such that the second coordinate stabilizes. We note that the choices $J_0 = \{0\}$ and $J_0 = \{2\}$ are Weyl equivalent in the horizontal algebra, but inequivalent in the affine algebra. They also are inequivalent as limiting choices. In this example, using the pseudo-Levi subalgebra classification scheme as a starting point for the semi-classical limits works, if only because another, inequivalent limit, does not stabilize.

5.4 Results Based on Numerics

In this subsection, we present results based on numerical analyses performed at finite coupling τ . The main strategy is to combine a random exploration of the parameter space with the requirement that vacua should form closed multiplets under S_3 and T dualities. Our numerics is in essence based on the FindMinimum procedure of Mathematica, applied to the logarithm of the positive definite real potential of the gauge theory. Let us first explain how these dualities can be implemented numerically on a vacuum that we know at large τ (by which we always mean the semi-classical regime $\tau \rightarrow i\infty$):

- T -duality is performed by taking the vacuum at large τ and changing continuously $\tau \mapsto \tau + 1$ on a straight line.
- For S_3 -duality, we first track the vacuum to the self-dual point $\tau_{sd} = i/\sqrt{3}$, then use the exact Langlands S_3 -duality formula (see later, equation (5.22)) to S_3 -dualize it, and finally bring it back to large τ .

Note that it is crucial that τ be large to T -dualize, because of potential points of monodromy at finite gauge coupling.

Finding the Vacua

Using our numerical procedure, it is easy to find the Higgs vacuum on the real axis; we label it H . Taking its S_3 -dual as explained above, one obtains the confining vacuum dubbed C_0 . When we T -dualize the confining vacua we obtain three more vacua, C_1, C_4, C_5 , for a quadruplet of confining vacua at large τ . More subtle is the following fact. Consider these vacua brought down to the self-dual value of the gauge coupling τ_{sd} . We call T_{sd} duality the operation

$$T_{sd} : \tau_{sd} = i/\sqrt{3} \mapsto \tau_{sd} + 1 \quad (5.19)$$

continuously along a straight line in the upper-half plane \mathcal{H} . If we apply this transformation to the confining vacuum C_0 , we find that we need to repeat it six times before falling on this confining vacuum once more. We thus find a sextuplet of T_{sd} -duality that we denote $(C_0, C_1, C_2, C_3, C_4, C_5)$. This indicates a point of monodromy that lies above the self-dual point.³⁰ The point of monodromy is located around $\tau_M \sim 1.440672920416i$, and all of these digits are significant. At the self-dual point, we can analytically check that S_3 -duality acts as $S_3(C_1) = C_4, S_3(C_2) = C_5$ and $S_3(C_3) = C_3$. Moreover, if we bring up the two extra

³⁰See [17] for a more gentle introduction to points of monodromy.

vacua (C_2, C_3) that complete the sextuplet to larger τ , they behave as a doublet under T -duality. These seven vacua obtained from the Higgs are represented on the right of figure 6.

In addition to these, we also find two extrema which are S_3 -duality and T -duality singlets, and also two S_3 -singlets (labelled J_1, J_2) which are T -dual (and T_{sd} -dual) to each other. This completes the duality web summarized in figure 6.

Identification with the Semi-classical Limits

We have obtained a total of eleven extrema, as expected from section 5.3. We can be more precise and match each T -multiplet with its corresponding limiting integrable system, using the value of the superpotential when necessary.

The singlets correspond to the 2- and 3-cycle semi-classical vacua, while the doublet of the duality group matches the semi-classical $J_0 = \{1\}$ extrema. The confining quadruplet is easily matched to the semi-classical solutions. The semi-classical origin of (C_2, C_3) is the choice $J_0 = \{2\}$. The numerical evidence we obtained for this last identification is limited to the first two coefficients in the superpotential (5.18).

Numerical values

Finally, let us provide a few concrete numbers of our simulations for easier reproducibility. The (z_1, z_2) positions of the numerical extrema are approximately given in the tables below, where the first entry is real part of z_1 and the second entry is the imaginary part of z_1 expressed in units of the purely imaginary value of τ .

Vacuum	Positions at $\frac{i}{\sqrt{3}}$	Positions at $\frac{5i}{2}$
H	{0.22754, 0., 0.03944, 0.}	{0.22738, 0., 0.03954, 0.}
H2	{0.16667, 0.5, 0.16667, 0.}	{0.16667, 0.5, 0.16667, 0.}
H3	{0.11111, 0.33333, 0.22222, 0.33333}	{0.11111, 0.33333, 0.22222, 0.33333}
C0	{0., 0.26698, 0., 0.41565}	{0., 0.26222, 0., 0.41795}
C1	{0.2727, 0.2679, 0.4226, 0.4103}	{0.25257, 0.2631, 0.41731, 0.41817}
C2	{0.5594, 0.3047, 0.8509, 0.4275}	{0.5, 0.33139, 0.83333, 0.42738}
C3	{0.86305, 0.33333, 1.26486, 0.46124}	{0.83812, 0.33333, 1.25239, 0.42829}
C4	{0.22607, 0.36197, 0.7085, 0.45614}	{0.16667, 0.40236, 0.66667, 0.4875}
C5	{0.60603, 0.39877, 0.18344, 0.47574}	{0.58591, 0.40356, 0.1686, 0.4884}
J1	{0.89497, 0.60134, 0.64682, 0.45821}	{0.87587, 0.58097, 0.62587, 0.41972}
J2	{0.98015, 0.20845, 0.56164, 0.73199}	{0.9592, 0.24694, 0.54253, 0.75236}

(5.20)

The superpotentials in these vacua are

Vacuum	Superpotential at $\frac{i}{\sqrt{3}}$	Superpotential at $\frac{5i}{2}$
H	271.5202972	256.6097930
H2	26.54254786	19.73924450
H3	26.54254786	19.73924450
C0	-218.4352014	-22.81452733
C1	42.47856497 - 33.32941024i	-19.56246124 - 2.892724428i
C2	10.60653076 + 33.32941024i	-9.869136924
C3	26.54254786	-9.869143660
C4	10.60653076 - 33.32941024i	-17.01826338
C5	42.47856497 + 33.32941024i	-19.56246124 + 2.892724428i
J1	26.54254786 + 13.36027231i	9.869700650 + 0.03957060700i
J2	26.54254786 - 13.36027231i	9.869700650 - 0.03957060700i

(5.21)

5.5 Langlands Duality and the Duality Diagram

Aside from the simply laced Lie algebras of A , D and E type, there are three more algebras that are mapped to themselves under Langlands duality. These are B_2 , G_2 and F_4 . The twisted elliptic integrable systems with appropriate couplings are indeed Langlands self-dual [17], namely, they permit the symmetry $S_\alpha : \tau \rightarrow -\frac{1}{\alpha\tau}$, where α is the ratio of the length squared of the long versus the short roots. The invariance under S_α translates into a relation involving the superpotentials evaluated at different positions X_i , including a shift. Explicitly, the fact that G_2 is invariant under $S_3 : \tau \rightarrow -\frac{1}{3\tau}$ duality reads

$$\mathcal{W}_{G_2,tw}(X_1, X_2; \tau) = \frac{1}{3\tau^2} \mathcal{W}_{G_2,tw} \left(\frac{X_1 + X_2}{3\tau}, \frac{2X_1 - X_2}{3\tau}; -\frac{1}{3\tau} \right) + 2\pi^2 [3E_2(3\tau) - E_2(\tau)] . \quad (5.22)$$

As was the case for the $so(5)$ integrable system (see [17]), the shift resulting from the S_α duality transformation can be identified with the superpotential in one of the vacua. The latter property allows for the realization of duality symmetries as permutations on the list of extremal superpotential values.

We have determined these permutations numerically (as reviewed above), and sum up the action of S_3 , T and T_{sd} in the diagram shown in figure 6. This diagram demonstrates the importance of specifying the path followed in the moduli space while performing a duality: note for instance that in the diagram, $S_3 T_{sd}$ has order 7 while the order of the more standard operation $S_3 T$ is 6, as a consequence of monodromies. In [17] one can find other examples of generalized duality groups that are generated by points of monodromy.

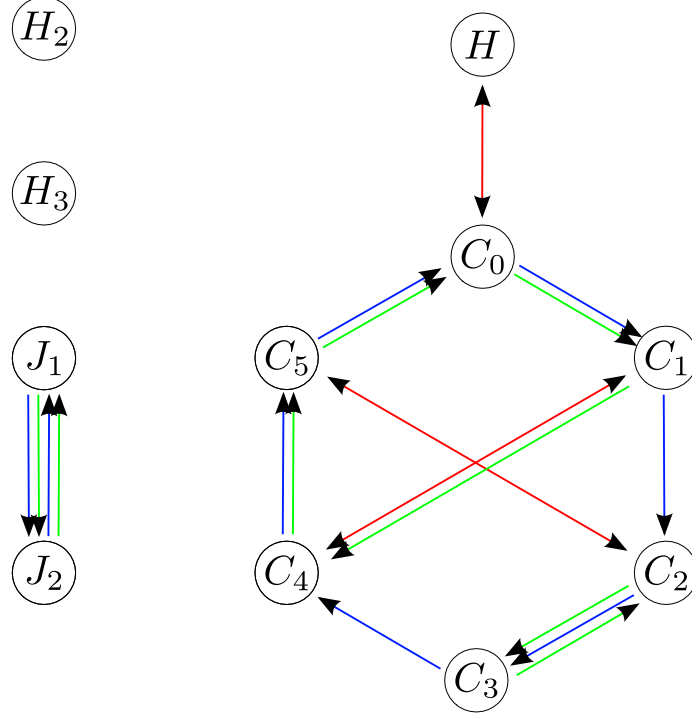


Figure 6: A diagram of dualities of G_2 , below the point of monodromy. The blue arrows represent the transformation $T_{sd} : \tau \rightarrow \tau + 1$ below the point of monodromy, while the green arrows represent the transformation $T : \tau \rightarrow \tau + 1$ above the point of monodromy. To identify vacua at different τ we use the convention that the branch cut is on the left of the monodromy point (\mathbb{R}_- direction). The red arrows indicate the action of S_3 -duality (5.22) at the self-dual point $\tau = \frac{i}{\sqrt{3}}$. The absence of a given arrow indicates invariance of the vacuum under the corresponding transformation. Note that the order of $S_3 T_{sd}$ is 7 while the order of $S_3 T$ is 6.

Finally, we make a few remarks on the exact values of the superpotential in a number of vacua. The superpotentials in the Higgs vacua with non-zero Wilson line are identical. They are equal to

$$\mathcal{W}_{H2} = 2\pi^2(\theta_3(q^2)\theta_3(q^6) + \theta_2(q^2)\theta_2(q^6))^2. \quad (5.23)$$

This is the theta series of the direct sum of 2 copies of a hexagonal lattice. It generates the (1-dimensional) space of modular forms of weight 2 for the congruence subgroup $\Gamma_0(3)$. Many further analytical statements can be made about the exact values of the superpotential. As an example, we have the closed form expression

$$26.54254786... = \frac{9\Gamma(\frac{1}{3})^6}{8 \times 2^{\frac{2}{3}}\pi^2}, \quad (5.24)$$

for this particular entry in table (5.21) of values of the superpotential. It will be interesting to classify the superpotential values into (vector valued) modular forms (potentially with non-analyticity in the upper half plane) of $\Gamma_0(3)$ or the full Hecke group.

6 The $so(5)$ Massless Branch

In this section, we tie up a loose end. In section 2 we analyzed semi-classical limits for the B_2 twisted elliptic integrable system, and we found a single massless branch of complex dimension one. We wish to characterize this branch more precisely, including at finite coupling τ . We also exhibit its duality and global properties in the different theories associated to the gauge algebra $so(5)$.

6.1 The Local Description of the Massless Branch

We show that a massless branch exists at all couplings by a brute force analysis. We postulate that the superpotential on the massless branch is equal to $e_1(q)$ (as we found in perturbation theory in section 2). We will also consider the two equations that follow from the fact that we are studying an extremum of the superpotential. These equations give rise to three constraint equations in terms of two unknowns, for a single massless branch of complex dimension one. This doubly overdetermined system will have a simple solution which is the description of the massless branch. Before we get to the simple end result, we plough through some elliptic function identities. Firstly, we recall the definition of the Weierstrass function evaluated at half-periods

$$\wp(\omega_i; \tau) = e_i(q) \quad (6.1)$$

and note that we have the equality

$$e_1(q) = -\frac{2\pi^2}{3}E_{2,2}(q) = -\frac{2\pi^2}{3}(E_2(q) - 2E_2(q^2)), \quad (6.2)$$

as well as the identities

$$\begin{aligned} \wp_2(z; \tau) &= 4\wp(2z; 2\tau) + e_1 \\ &= -e_1 + \frac{1}{4} \left(\frac{\wp'(z; \tau)}{\wp(z; \tau) - e_1} \right)^2 \\ \wp_2'(z; \tau) &= 8\wp'(2z; 2\tau). \end{aligned} \quad (6.3)$$

We again describe the superpotential and its derivatives algebraically using the variables³¹

$$\mathcal{X}_i = \wp(z_i; \tau) \quad \text{and} \quad \mathcal{Y}_i = \wp'(z_i; \tau), \quad (6.4)$$

for $i = 1, 2$. The value of the superpotential, $\mathcal{W}_{B_2, tw}(Z; \tau) = e_1(\tau)$, translates into the equation

$$2 \left(\frac{\mathcal{Y}_1 - \mathcal{Y}_2}{\mathcal{X}_1 - \mathcal{X}_2} \right)^2 + 2 \left(\frac{\mathcal{Y}_1 + \mathcal{Y}_2}{\mathcal{X}_1 - \mathcal{X}_2} \right)^2 + \left(\frac{\mathcal{Y}_1}{\mathcal{X}_1 - e_1} \right)^2 + \left(\frac{\mathcal{Y}_2}{\mathcal{X}_2 - e_1} \right)^2 = 16(e_1 + \mathcal{X}_1 + \mathcal{X}_2). \quad (6.5)$$

³¹These variables describe faithfully the vacua of the $SO(5)_+$ theory, by which we mean that for any $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2) \in \mathbb{C}^4$ there is exactly one vacuum of the $SO(5)_+$ theory that satisfies (6.4). In the $Spin(5)$ and $SO(5)_-$ theories there are two such vacua, namely (z_1, z_2) and $(z_1 + 2\omega_2, z_2 + 2\omega_2)$. Moreover, both in the $Spin(5)$ and in the $SO(5)_-$ theory (6.4) is not a well defined functional of a given vacuum because of the identification $(z_1, z_2) \sim (z_1 + \omega_1, z_2 + \omega_1)$ in $Spin(5)$ and $(z_1, z_2) \sim (z_1 + \omega_1 + 2\omega_2, z_2 + \omega_1)$ in $SO(5)_-$. These subtleties will be taken care of in subsection 6.3.

We also need the addition formula for the derivative of the Weierstrass function:

$$\wp'(u+v) = \frac{\left[\frac{1}{2}g_3(\wp'(u) - \wp'(v)) + (\wp'(v)\wp(u)^2 + \frac{1}{4}g_2\wp'(u))(\wp(u) + 3\wp(v))\right] - [u \leftrightarrow v]}{(\wp(u) - \wp(v))^3} \quad (6.6)$$

It is convenient to write this in a more symmetric way, where all the derivatives are isolated on the left-hand side :

$$\frac{\wp'(u+v) + \wp'(u-v)}{\wp'(u)} = \frac{g_2(\wp(u) + 3\wp(v)) + 4g_3 - 4\wp(v)^2(3\wp(u) + \wp(v))}{2(\wp(u) - \wp(v))^3}, \quad (6.7)$$

which we can use to express the derivative of the twisted Weierstrass function as

$$\frac{\wp'_2(u)}{\wp'(u)} = \frac{8e_1^3 - 3e_1(g_2 - 4\wp(u)^2) - \wp(u)(g_2 + 4\wp(u)^2) - 4g_3}{4(e_1 - \wp(u))^3}. \quad (6.8)$$

Using these relations, the three equations describing the massless branch become

$$\begin{aligned} \frac{4(4\mathcal{X}_1^3 + 4\mathcal{X}_2^3 - g_2(\mathcal{X}_1 + \mathcal{X}_2) - 2g_3)}{(\mathcal{X}_1 - \mathcal{X}_2)^2} + \frac{4\mathcal{X}_1^3 - g_2\mathcal{X}_1 - g_3}{(\mathcal{X}_1 - e_1)^2} + \frac{4\mathcal{X}_2^3 - g_2\mathcal{X}_2 - g_3}{(\mathcal{X}_2 - e_1)^2} - 16(e_1 + \mathcal{X}_1 + \mathcal{X}_2) &= 0 \\ \frac{g_2(\mathcal{X}_1 + 3\mathcal{X}_2) + 4g_3 - 4\mathcal{X}_2^2(3\mathcal{X}_1 + \mathcal{X}_2)}{(\mathcal{X}_1 - \mathcal{X}_2)^3} + \frac{8e_1^3 - 3e_1(g_2 - 4\mathcal{X}_1^2) - \mathcal{X}_1(g_2 + 4\mathcal{X}_1^2) - 4g_3}{4(e_1 - \mathcal{X}_1)^3} &= 0 \\ \frac{g_2(\mathcal{X}_2 + 3\mathcal{X}_1) + 4g_3 - 4\mathcal{X}_1^2(3\mathcal{X}_2 + \mathcal{X}_1)}{(\mathcal{X}_2 - \mathcal{X}_1)^3} + \frac{8e_1^3 - 3e_1(g_2 - 4\mathcal{X}_2^2) - \mathcal{X}_2(g_2 + 4\mathcal{X}_2^2) - 4g_3}{4(e_1 - \mathcal{X}_2)^3} &= 0. \end{aligned}$$

Finally, we express the Eisenstein series g_2 and g_3 of weight 4 and 6 in terms of the half-period values e_i using the relations $g_2 = 2(e_1^2 + e_2^2 + e_3^2)$ and $g_3 = 4e_1e_2e_3$ to obtain

$$\frac{(2e_1 - \mathcal{X}_1 - \mathcal{X}_2)^2}{(e_1 - \mathcal{X}_1)(e_1 - \mathcal{X}_2)(\mathcal{X}_1 - \mathcal{X}_2)^2} P_3(\mathcal{X}_1, \mathcal{X}_2, e_1, e_2) = 0 \quad (6.9)$$

$$\frac{(2e_1 - \mathcal{X}_1 - \mathcal{X}_2)}{(e_1 - \mathcal{X}_1)^2(\mathcal{X}_1 - \mathcal{X}_2)^3} P_4(\mathcal{X}_1, \mathcal{X}_2, e_1, e_2) = 0 \quad (6.10)$$

$$\frac{(2e_1 - \mathcal{X}_1 - \mathcal{X}_2)}{(e_1 - \mathcal{X}_2)^2(\mathcal{X}_2 - \mathcal{X}_1)^3} P_4(\mathcal{X}_2, \mathcal{X}_1, e_1, e_2) = 0, \quad (6.11)$$

where P_3 and P_4 are homogeneous polynomials of degree 3 and 4 respectively. We see that $\mathcal{X}_1 + \mathcal{X}_2 = 2e_1$ is a sufficient condition to be on the massless branch of vacua. Restricting to these solutions – except at special points in the space of couplings, these are the only solutions –, we can parametrize the line by a single complex number $\lambda \in \mathbb{C}^*$ as

$$\mathcal{X}_1 = e_1(\tau) + \lambda, \quad \mathcal{X}_2 = e_1(\tau) - \lambda. \quad (6.12)$$

6.2 Duality and the Massless Branch

T-duality manifestly leaves the description of the massless branch in terms of the elliptic curve variables invariant, as can be seen from equation (6.12). We analyze Langlands S_2 duality next. In the notation of equation (2.1), the duality formula for $so(5)$ reads [17]

$$\mathcal{W}_{B_2, tw}(z_1, z_2, \tau) = \frac{1}{2\tau^2} \mathcal{W}_{B_2, tw}\left(\frac{z_1 + z_2}{2\tau}, \frac{z_1 - z_2}{2\tau}, -\frac{1}{2\tau}\right) + 2e_1(\tau). \quad (6.13)$$

Using the identity

$$e_1\left(-\frac{1}{2\tau}\right) = -2\tau^2 e_1(\tau), \quad (6.14)$$

it can be written in the more symmetric form

$$\mathcal{W}_{B_2,tw}(z_1, z_2, \tau) - e_1(\tau) = \frac{1}{2\tau^2} \left[\mathcal{W}_{B_2,tw}\left(\frac{z_1 + z_2}{2\tau}, \frac{z_1 - z_2}{2\tau}, -\frac{1}{2\tau}\right) - e_1\left(-\frac{1}{2\tau}\right) \right]. \quad (6.15)$$

We define the dual elliptic curve variables

$$\mathcal{X}'_1 = \wp\left(\frac{z_1 + z_2}{2\tau}; -\frac{1}{2\tau}\right) \quad (6.16)$$

$$\mathcal{X}'_2 = \wp\left(\frac{z_1 - z_2}{2\tau}; -\frac{1}{2\tau}\right) \quad (6.17)$$

$$\mathcal{Y}'_1 = \wp'\left(\frac{z_1 + z_2}{2\tau}; -\frac{1}{2\tau}\right) \quad (6.18)$$

$$\mathcal{Y}'_2 = \wp'\left(\frac{z_1 - z_2}{2\tau}; -\frac{1}{2\tau}\right), \quad (6.19)$$

which are related to the original elliptic curve variables (6.4) by

$$-e_1(\tau) - \mathcal{X}_1 - \mathcal{X}_2 + \frac{1}{4} \left(\frac{\mathcal{Y}_1 - \mathcal{Y}_2}{\mathcal{X}_1 - \mathcal{X}_2} \right)^2 = \frac{1}{16\tau^2} \left(\frac{\mathcal{Y}'_1}{\mathcal{X}'_1 - e_1(-\frac{1}{2\tau})} \right)^2 \quad (6.20)$$

$$-e_1(\tau) - \mathcal{X}_1 - \mathcal{X}_2 + \frac{1}{4} \left(\frac{\mathcal{Y}_1 + \mathcal{Y}_2}{\mathcal{X}_1 - \mathcal{X}_2} \right)^2 = \frac{1}{16\tau^2} \left(\frac{\mathcal{Y}'_2}{\mathcal{X}'_2 - e_1(-\frac{1}{2\tau})} \right)^2. \quad (6.21)$$

The sum of these relations is

$$-2e_1(\tau) - 2\mathcal{X}_1 - 2\mathcal{X}_2 + \frac{\mathcal{Y}_1^2 + \mathcal{Y}_2^2}{2(\mathcal{X}_1 - \mathcal{X}_2)^2} = \frac{1}{16\tau^2} \left(\frac{\mathcal{Y}'_1}{\mathcal{X}'_1 - e_1(-\frac{1}{2\tau})} \right)^2 + \frac{1}{16\tau^2} \left(\frac{\mathcal{Y}'_2}{\mathcal{X}'_2 - e_1(-\frac{1}{2\tau})} \right)^2.$$

After these preparations, we will now show that if we choose a point on the dual of the massless branch, namely a point satisfying the equation $\mathcal{X}'_1 + \mathcal{X}'_2 = 2e_1(-\frac{1}{2\tau})$, that this is consistent with the original variables lying on the original massless branch. Indeed, this equality implies that the sum of the relations becomes

$$2(e_1(\tau) + \mathcal{X}_1 + \mathcal{X}_2) - \frac{\mathcal{Y}_1^2 + \mathcal{Y}_2^2}{2(\mathcal{X}_1 - \mathcal{X}_2)^2} + \frac{(\mathcal{Y}'_1)^2 + (\mathcal{Y}'_2)^2}{4\tau^2(\mathcal{X}'_1 - \mathcal{X}'_2)^2} = 0. \quad (6.22)$$

Taking into account the elliptic curve equation, we can simplify this to

$$\frac{(\mathcal{Y}'_1)^2 + (\mathcal{Y}'_2)^2}{4\tau^2(\mathcal{X}'_1 - \mathcal{X}'_2)^2} = \frac{3}{4\tau^2}(\mathcal{X}'_1 + \mathcal{X}'_2) = \frac{3}{2\tau^2}e_1\left(-\frac{1}{2\tau}\right) = -3e_1(\tau), \quad (6.23)$$

and we end up with

$$\frac{(-2e_1 + \mathcal{X}_1 + \mathcal{X}_2)(-2e_1e_2 + e_1\mathcal{X}_1 + e_1\mathcal{X}_2 - 2e_2^2 + 2\mathcal{X}_1\mathcal{X}_2)}{(\mathcal{X}_1 - \mathcal{X}_2)^2} = 0. \quad (6.24)$$

Finally, we see that this equality is implied by the original point being on the original branch $-2e_1 + \mathcal{X}_1 + \mathcal{X}_2 = 0$, and we have therefore obtained a non-trivial check of the statement that the massless branch is invariant under S_2 -duality.

6.3 The Moduli Space of Vacua for the Different Gauge Theories

In this subsection we obtain the global structure of the moduli space of massless vacua for the different theories with $so(5)$ gauge algebra, taking into account various discrete identifications. We also perform a consistency check on these global properties by showing how S_2 -duality acts on these moduli spaces, thus completing the results in subsection 2.3.

We wish to characterize the branch by extracting the positions z_i from the elliptic curve variables (6.4), and this should be done up to Weyl equivalence. The Weyl group is generated by two reflections: the reflection about α_1 leads to the identification

$$(\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2) \equiv (\mathcal{X}_2, -\mathcal{Y}_2, \mathcal{X}_1, -\mathcal{Y}_1), \quad (6.25)$$

while the reflection about α_2 gives the identification

$$(\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2) \equiv (\mathcal{X}_1, -\mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2). \quad (6.26)$$

This shows that the sign of the variables \mathcal{Y}_i is irrelevant, and we no longer need to keep track of them. The Weyl symmetry therefore implies that we can study the manifold described by the variables $(\mathcal{X}_1, \mathcal{X}_2)$ subject to the identification $(\mathcal{X}_1, \mathcal{X}_2) \equiv (\mathcal{X}_2, \mathcal{X}_1)$. The branch of massless vacua of the $SO(5)_+$ theory, for which there is no additional identification, is described by $\lambda \in \mathbb{C}^*/\mathbb{Z}_2$, where the \mathbb{Z}_2 action is $\lambda \leftrightarrow -\lambda$. This is a sphere with two excised points.

In the $SO(5)_-$ theory we have the additional identification $(z_1, z_2) \equiv (z_1 + \omega_1 + 2\omega_2, z_2 + \omega_1)$. On the manifold parametrized by λ it corresponds to $\lambda \equiv \lambda' = \pi^4 \theta_4^8(2\tau)/\lambda$. This follows from the observation that if $\wp(z_1) = e_1 + \lambda$, then

$$\wp(z_1 + \omega_1) = -e_1 - (e_1 + \lambda) + \frac{\wp'(z_1)^2}{4\lambda^2} \quad (6.27)$$

$$= e_1 + \frac{3e_1^2 - \frac{1}{4}g_2}{\lambda} + \frac{4e_1^3 - g_2e_1 - g_3}{\lambda^2} \quad (6.28)$$

$$= e_1 + \lambda', \quad (6.29)$$

and similarly if $\wp(z_2) = e_1 - \lambda$ then $\wp(z_2 + \omega_1) = e_1 - \lambda'$. Note that the function $\theta_4(\tau)$ doesn't vanish on the upper-half plane,³² so that $\lambda \mapsto \lambda'$ is a well-defined involution everywhere in the moduli space.

For a given $\lambda \in \mathbb{C}^*$, the $SO(5)_-$ theory has two non-equivalent vacua (z_1, z_2) and $(z_1 + 2\omega_2, z_2)$ which correspond to this λ . These two vacua are respectively equivalent to $(z_1 + \omega_1 + 2\omega_2, z_2 + \omega_1)$ and $(z_1 + \omega_1, z_2 + \omega_1)$, which are associated to the same λ' . Therefore the branch of massless vacua is a double cover of $\mathbb{C}^*/\mathbb{Z}_2$.

As for the $Spin(5)$ theory, we also need to take a double cover of the quotiented sphere $\mathbb{C}^*/\mathbb{Z}_2$. For a generic $\lambda \in \mathbb{C}^*$, the two vacua λ and λ' in the $SO(5)_+$ theory are inequivalent. They are mapped by S_2 to inequivalent vacua that share the same dual λ_D , or equivalently the same λ'_D . We see that S_2 -duality cancels the cover and the quotient to recover the manifold for $SO(5)_+$ which is just \mathbb{C}^* .

³²The zeros of $\theta_4(z, \tau)$ are given by $z = n + (m + 1/2)\tau$ with $n, m \in \mathbb{Z}$

7 Conclusions

In this paper, we further explored the vacuum structure and duality properties of $\mathcal{N} = 1^*$ gauge theories. We also found new phenomena in elliptic integrable systems. In the latter, we identified limits which exhibit a staircase structure – degrees of freedom are fixed at various powers of the modular parameter q of the integrable system. This has a counterpart in the instanton effects responsible for fixing vacuum expectation values in the gauge theory. For the gauge theory compactified on the circle, we clarified multiple phenomena. We have exhibited massive vacua of $\mathcal{N} = 1^*$ gauge theories associated to discrete component groups of nilpotent orbits. We also found vacua that become massive due to discrete Wilson lines. Moreover, we started the study of branches of massless vacua of the $\mathcal{N} = 1^*$ theory. For the $su(3)$ gauge algebra, we gave the equation for the massless branch, and identified the (Argyres-Douglas) singularity. We moreover plunged into the elliptic function theory that enters the exact description of the corresponding equilibrium positions of the elliptic integrable systems, and their duality properties. We thus provided a physical application (and extension) of the Eichler-Zagier formulas. Moreover, we laid bare the massless branch of vacua for the theory with $so(5)$ gauge group. Our analysis invoked a combination of the rich semi-classical limits of elliptic integrable systems, numerical data, modular forms and elliptic function theory.

We have walked into a field which is littered with interesting open problems. Let us enumerate just a few.

- Count and characterize massive vacua and massless branches (of differing dimension) of vacua of $\mathcal{N} = 1^*$ gauge theories on $\mathbb{R}^{2,1} \times S^1$, or $\mathcal{N} = 1$ theories in general.
- Compute the duality diagram for all the vacua.
- Understand the (vector valued) modular objects with monodromies in the interior and on the boundary of the fundamental domain that naturally appear as equilibrium positions, as well as those that appear as extremal values of the potential of elliptic integrable systems.
- Analyze the desingularization of the effective superpotential when it develops monodromies.
- Classify complex extrema of integrable systems.
- Compute all possible staircase limits of elliptic integrable systems, as well as their extrema, and generalize these limits, for instance to integrable systems with spin.
- Identify all massive and massless vacua on $\mathbb{R}^{2,1} \times S^1$ from (a generalization to the compactified theory of) the Seiberg-Witten curve of the $\mathcal{N} = 2^*$ theory.
- Investigate the relation to the geometric Langlands program and nilpotent orbit theory as applied to defect theories.

- Complete the analysis of global aspects of the theory, including the global choice of gauge group and the spectrum of line operators.

We hope to revisit this field fruitfully in the future.

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A The Lie Algebra and the Group G_2

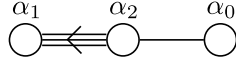


Figure 7: The Dynkin diagram of the affine algebra $G_2^{(1)}$.

The Cartan matrix of G_2 is

$$A_{G_2} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \quad (\text{A.1})$$

The adjoint representation has dimension 14. We can represent the root and weight system in terms of three linearly dependent vectors $\epsilon_{1,2,3}$ satisfying $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. The basis we use is (ϵ_1, ϵ_2) . It satisfies the relations $|\epsilon_1|^2 = |\epsilon_2|^2 = \frac{2}{3}$ and $\epsilon_1 \cdot \epsilon_2 = -\frac{1}{3}$. (In fact, analogous relations hold for all ϵ_i .) The roots are given by $\epsilon_i \pm \epsilon_j$ ($i \neq j$) and $\pm\epsilon_i$ for a total of 12. We have 6 positive roots. There are three short positive roots, ϵ_1 , $\alpha_1 = -\epsilon_2$ and $\epsilon_1 + \epsilon_2$, with length squared $\frac{2}{3}$ and three long roots, $\alpha_2 = \epsilon_1 + 2\epsilon_2$, $2\epsilon_1 + \epsilon_2$ and $\epsilon_1 - \epsilon_2$ with length squared 2. The ratio of lengths squared is equal to three. The simple roots can be represented as $\alpha_1 = -\epsilon_2$ and $\alpha_2 = \epsilon_2 - \epsilon_3$. The highest root is $\epsilon_1 - \epsilon_3$, and it is also equal to the second fundamental weight. The first fundamental weight is ϵ_1 . The weight lattice is spanned by the ϵ_i .

The co-roots $\alpha_1^\vee = -3\epsilon_2$ and $\alpha_2^\vee = \epsilon_1 + 2\epsilon_2$ have length squared equal to 6 and 2 respectively. We deduce the fundamental co-weights $\pi_1^\vee = 3\epsilon_1$ and $\pi_2^\vee = 2\epsilon_1 + \epsilon_2$ and the fundamental weights $\pi_1 = \epsilon_1$ and $\pi_2 = 2\epsilon_1 + \epsilon_2$. Finally, the Weyl group has 12 elements, it is precisely

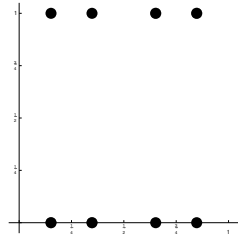
$$\{(r_1 r_2)^n (r_1)^\epsilon | 0 \leq n \leq 5 \text{ and } \epsilon = 0, 1\}, \quad (\text{A.2})$$

where r_i are simple Weyl reflections. One of those elements, $(r_1 r_2)^2 r_1$, exchanges ϵ_1 and ϵ_2 , meaning that in the bulk of the paper, extrema with z_1 and z_2 exchanged are considered equivalent. A global sign flip is also allowed by $(r_1 r_2)^3 = -1$. Finally $(r_1 r_2)^3 r_1$ acts as

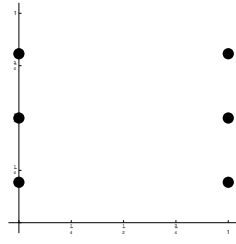
$$\begin{cases} \epsilon_1 \rightarrow \epsilon_1 + \epsilon_2 \\ \epsilon_2 \rightarrow -\epsilon_2. \end{cases} \quad (\text{A.3})$$

The final litany of useful facts includes that both the center of G_2 and its group of outer automorphisms are trivial. The algebra G_2 is its own Langlands dual. The dual of the (non-twisted) affine algebra $G_2^{(1)}$ on the other hand is $\left(G_2^{(1)}\right)^\vee = D_4^{(3)}$. This last algebra has two short simple roots and one long simple root whose length squared is three times larger. The co-marks of $g_2^{(1)}$ are $(1, 2, 1)$.

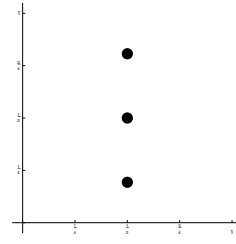
B Representations of the Vacua for B_2 Theories



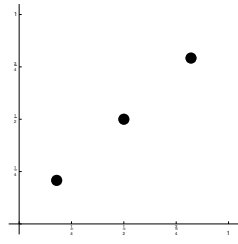
Extremum 1



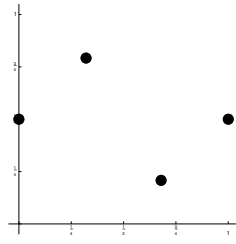
Extremum 2



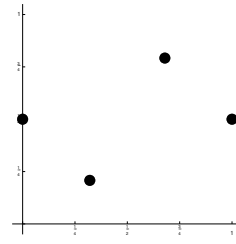
Extremum 2b



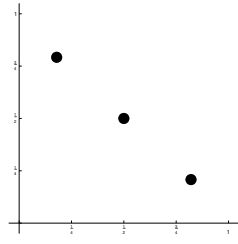
Extremum 3



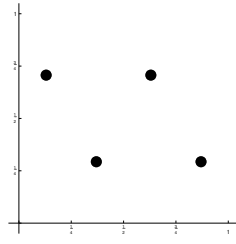
Extremum 3b



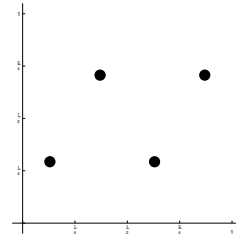
Extremum 4



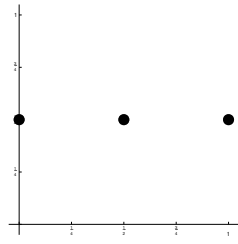
Extremum 4b



Extremum 5

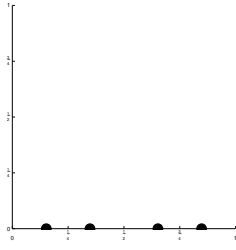


Extremum 6

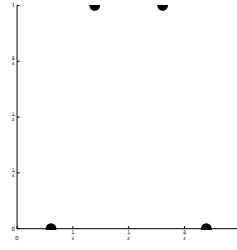


Extremum 7

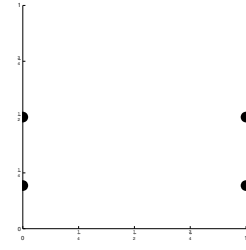
Figure 8: The extremal positions of the vacua for the $SO(5)_+$ theory.



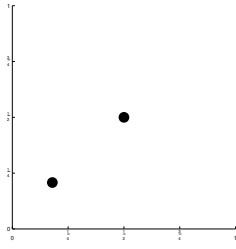
Extremum 1



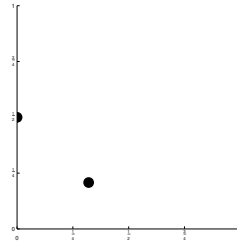
Extremum 1b



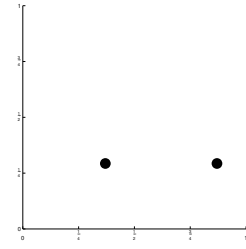
Extremum 2



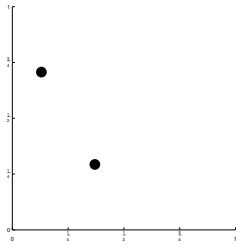
Extremum 3



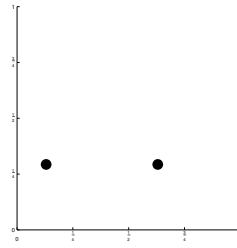
Extremum 4



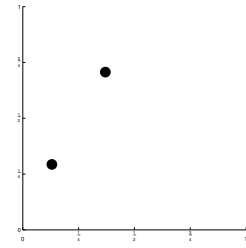
Extremum 5



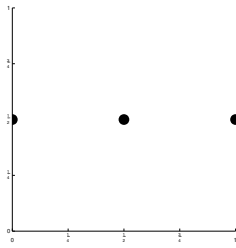
Extremum 5b



Extremum 6



Extremum 6b



Extremum 7

Figure 9: The extremal positions of the vacua for the $Spin(5)$ theory.

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